

Tips for the first round of the Georg Mohr Contest

Number theory

Number theory is about the integers, and especially about when an integer divides another integer. Therefore, the prime numbers have a very central role in number theory, which we'll be looking at soon.

Here, we present ideas on how to solve number theory problems. It is not a theoretical exposition, but instead focused on exemplifying some central principles and ideas about how to, for example, use divisibility rules and prime factor decomposition. To succeed in the first round of the Georg Mohr Contest, the most important thing is to practice a lot of exercises. Therefore, this document contains a collection of problems, many of them previously used as first round Georg Mohr questions.

In the first round of the Georg Mohr Contest, the first ten problems are multiple choice questions with five answers to choose from, while the last ten problems have answers in the form of positive integers. The problems here are a mix of these two kinds. There are solution sketches for all problems in the back.

Divisor The number 3 divides 99 because $99 = 3 \cdot 33$. In general, if d and n are integers, we say that d divides n if there exists an integer q such that

$$n = q \cdot d.$$

We also say that d is a *divisor* of n, or that n is *divisible by* d.

Prime numbers A positive integer larger than 1 is called a prime number if the only positive divisors of the number is 1 and the number itself. The first ten primes are

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29.$$

Divisibility rules

There are many rules about divisibility. Children quickly learn that 2 divides a number when its last digit is even, and that 5 divides a number when its last digit is 0 or 5, but there are more rules that aren't as well known or as intuitive.

The 3-rule A number is divisible by 3 if and only if its digital sum is divisible by 3. The digital sum of a number is the sum of its digits.

For example: 3 divides 1911114 because 3 divides the digital sum $1+9+1+1+1+1+4 = 18$.

The 9-rule A number is divisible by 9 if and only it its digital sum is divisible by 9.

For example: 9 divides 1911114 because 9 divides the digital sum 18.

The 11-rule A number is divisible by 9 if and only it its alternating digital sum is divisible by 9. The alternating digital sum of a number is calculated by alternating between adding and subtracting the digits of the number.

For example: 9 divides 918592950 because its alternating digital sum is $9-1+8-5+9-2+9-5+0 = 22$

Product of primes A product $p \cdot q$ of two distinct primes p and q divides a number if both p and q divides it.

For example, if we want to check if a number is divisible by $6 = 2 \cdot 3$, we can instead check if it is divisible by both 2 and 3.

Product For all integers n, m, a and b, if we know that n divides a and m divides b, then $n \cdot m$ divides $a \cdot b$.

For example, $27 = 3 \cdot 9$ divides $6 \cdot 1971$ because 3 divides 6, and 9 divides 1971.



Problem 1. How many of the following numbers are divisible by 9?

$$9393939, \quad 111111111, \quad 123456, \quad 123456 \cdot 15, \quad 99 \cdot 111$$

Problem 2. Which of the following numbers is not divisible by 3?

A) $333+45519$ B) $12445+54421$ C) $2777+7222$
D) $9416+7864$ E) $212121+414141$

(*Georg Mohr 2006, problem 3*)

Problem 3. Georg thinks of an integer. First he multiplies the number by 5 and subtracts 7 from the result. Then he multiplies the new result by 9 and obtains the number x. What can one say about x with certainty?

A) x is odd B) x is even C) 3 divides x D) 5 divides x E) 7 divides x

(*Georg Mohr 2016, problem 2*)

Problem 4. What is the smallest integer consisting of only the digits 1, 2 and 3, which contains each of the three digits at least once, and which 3 does not divide?

(*Georg Mohr 2015, problem 12*)

Problem 5. Which of the following problems cannot be solved? Write a 5-digit number composed of all the digits 1, 2, 3, 4 and 5 so that the number is divisible by

A) 4 B) 5 C) 6 D) 11 E) 125

(*Georg Mohr 2007, problem 10*)

Problem 6. Peter has written two positive integers x and y on a piece of paper which he shows to his friends A, B, C, D, and E. The five friends state the following:

A) $x+y$ is odd B) 3 divides y C) $y=2x$
D) $x \cdot y$ is odd E) $x<y$

Peter tells us that exactly one of the five friends lies. Who?

(*Georg Mohr 2014, problem 8*)

Problem 7. If m is an even number, and n is divisible by 6, which of the following numbers is then with certainty divisible by 4?

A) $m+n$ B) $mn-m$ C) m^2+n D) $m(m+n)$ E) $(m+1)n$

(*Georg Mohr 2009, problem 10*)

Problem 8. There are 12 presents numbered 1 through 12 on a table. Arne's favourite number is 2, Bent's favourite number is 3, Carl's favourite number is 4, Dora's favourite number is 5, and Ella's favourite number is 6. Arne, Bent, Carl, Dora and Ella walk up to the table in some order and receive the presents (from among those left) whose number their favourite number divides. It turns out that the last one gets two presents. Who is the last one?

A) Arne B) Bent C) Carl D) Dora E) Ella

(*Georg Mohr 2018, problem 9*)

Prime factor decomposition To factor a number, we write it as a product of primes. For example, the prime factor decomposition of 60 is $2^2 \cdot 3 \cdot 5$, the prime factor decomposition of 6 is $2 \cdot 3$ and the prime factor decomposition of 13 is simply 13. Every positive integer larger than 1 has a prime factor decomposition, and it is unique up to the order of the factors.

The prime factor decomposition of a number tells us something about which numbers divide it. For example, you can immediately see that 7 divides $2^8 \cdot 5^2 \cdot 7^9 \cdot 19$, and that 13 and 17 do not. You can also see that $25 = 5^2$ and $70 = 2 \cdot 5 \cdot 7$ are divisors.

If you need to find every divisor of 30, you can look at the prime factor decomposition $2 \cdot 3 \cdot 5$ and quickly determine that the complete list of divisors is 1, 2, 3, 5, $2 \cdot 3$, $2 \cdot 5$, $3 \cdot 5$ and $2 \cdot 3 \cdot 5$. Similarly, the list of divisors of $16 = 2^4$ is $1, 2, 2^2, 2^3, 2^4$.

Prime numbers function as building blocks that compose all positive integers, and for that reason they're central to any work with integers.



Problem 9. Which of the following numbers do not divide $2^9 \cdot 3 \cdot 7 \cdot 11 \cdot 13^4$?

A) 6 B) 14 C) 39 D) 121 E) 64

Problem 10. A man has his entire fortune distributed in solid fur bags, each containing 49 gold coins. One day he redistributes the fortune, so that it lies instead in 35 boxes with the same number of coins in each box. There are various rumours telling how many gold coins his fortune consists of. Which of the five rumours A, B, C, D, and E below may be true?

A) 70705 B) 6335 C) 2450 D) 70707 E) 4931

(*Georg Mohr 2014, problem 9*)

Problem 11. The product of four distinct positive integers is 1001. What is their sum?

Problem 12. The numbers from 1 to n are shown on a long list. From the list, all numbers divisible by 3 and all numbers divisible by 5 are now removed. Afterwards, 240 numbers remain on the list. You are given that both 3 and 5 divide the number n. What is n?

(*Georg Mohr 2014, problem 15*)

Problem 13. Integers a, b and c are known to satisfy $a^2b^3c^4 = -27$ How many possibilities are there for the value of c?

A) 0 B) 1 C) 2 D) 3 E) 4

(*Georg Mohr 2016, problem 7*)

Problem 14. Let p be a prime. How many distinct positive integers divide p^{10}?

Last digit

Last digit of a sum of two numbers To determine the last digit of a sum of two integers, you can look at the last digit of each of the integers, add them together, and find the last digit of this sum. For example, the last digit of $947568 + 3749663$ is 1 because $8 + 3 = 11$.

Last digit of a product of two numbers Similarly, to determine the last digit of a product of two integers, we can again look at the last digit of each factor and find the last digit of their product. For example, the last digit of $9328578 \cdot 2648658$ is 4, because $8 \cdot 8 = 64$.

Example We want to determine the last digit of 2007^{2007}. As the last digit of a number only depends on the last digits of the factors, this means we can instead find the last digit of 7^{2007}. This is still an enormous number that is impossible to compute, so we need to further simplify the problem.

If we look at $7^2 = 49$, its last digit is 9. We can use that to find the last digit of $7^3 = 7 \cdot 7^2$, because that has to be the same as the last digit of $7 \cdot 9 = 63$, which is 3. Similarly, the last digit of $7^4 = 7 \cdot 7^3$ must be the same as the last digit of $7 \cdot 3 = 21$, which is 1. The last digit of $7^5 = 7 \cdot 7^4$is equal to the last digit of $7 \cdot 1 = 7$. The last digit of 7^6 is equal to the last digit of $7 \cdot 7 = 49$, and so on. Because the last digit of 7^n only depends on the last digit of 7^{n-1} multiplied by 7, we observe a repeating pattern in the last digits of the powers:

Power	7	7^2	7^3	7^4	7^5	7^6	7^7	7^8	7^9	...
Last digit	7	9	3	1	7	9	3	1	7	...

The last digits of the powers repeat with a period of 4. This makes it easy to find the last digit of 7^{2007}. Because $2007 = 4 \cdot 501 + 3$, we can see that 7^{2007} must have the same last digit as 7^3, which is 3.

Problem 15. What is the last digit of 74851^3?



Problem 16. What is the last digit of 9^{2013}?

Problem 17. Which of the following numbers does not end in 5?

A) $5+15+25+35+45$ B) $5\cdot 15\cdot 25\cdot 35\cdot 45$ C) 19^2-16^2
D) $\frac{20\cdot 35\cdot 9}{4\cdot 15}$ E) $25^{25}-25^5$

(*Georg Mohr 2011, problem 11*)

Problem 18. The number 2000^{1000} ends in a lot of zeroes. What is the last non-zero digit?

Problem 19. A person has calculated n^{16} for all values of n from 1 to 100, and carefully written down the last digit of each result on a piece of paper. How many distinct digits appear on this piece of paper?

More about divisibility

Some basic divisibility rules about sums and differences:

Sum If an integer d divides two integers n and m,then d also divides their sum $n+m$. For example, if $a=3b+12c$, then 3 divides a because 3 divides $3b$, 3 divides $12c$, and therefore also the sum $3b+12c$.

Difference If an integer$d$ divides two integers n and m, then d also divides their difference $n-m$.

Example For two positive integers a and b, if we know that

$$a+5b=b^3,$$

we can deduce that b must divide a because b divides $5b$ and b^3, and therefore also their difference $a=b^3-5b$.

Example We want to find all positive integers a such that $a+3$ divides $4a+16$. We rewrite $4a+16$:

$$4a+16=4(a+3)+4.$$

In order for $a+3$ to divide $4a+16$, it must also divide $4=4a+16-4(a+3)$. The only positive integer a where $a+3$ divides 4 is $a=1$.

Problem 20. The positive integer n has the property that $4n+1$ divides $12n+10$. What is the digital sum of n?

A) 6 B) 7 C) 11 D) There is more than one option
E) There is no n with this property

Problem 21. Which of the following numbers is not divisible by 18?

A) 540 B) $1818+7216$ C) $36\cdot 4226$
D) $2007\cdot 2008\cdot 2009$ E) $9\cdot 888-8\cdot 999$

(*Georg Mohr 2008, problem 5*)

Problem 22. How many positive integers n exist with the property that $2n+1$ divides $8n+69$?

Problem 23. Given two integers n and m we are told that $n+m=n^2$. From this, you can deduce that

A) n divides m B) m divides n C) n and m are odd
D) n and m are even E) $n=m$



Extra challenges

Problem 24. Peter and his friends have 100 large chocolate frogs. They share them equally among themselves. Peter immediately eats one of his frogs and then one more, but at some point he feels sick and stops eating more. He computes that he has eaten exactly one quarter of his frogs. How many frogs does Peter have left?

(*Georg Mohr 2017, problem 18*)

Problem 25. A box contains sixty notes numbered from 1 to 60. Another box also contains sixty notes numbered from 1 to 60. Sixty people each draw a note at random from each box and multiply their two numbers. If the result is divisible by 6, the person gets a soft drink. How many soft drinks will at most be needed?

(*Georg Mohr 2015, problem 19*)

Problem 26. You are given that exactly one of the numbers 14, 15, 16, 20, 25 is a solution to the equation

$$5x^5+3x^3+x=3604065+4x^4+2x^2.$$

Which of the numbers is it?

A) 14 B) 15 C) 16 D) 20 E) 25

(*Georg Mohr 2014, problem 10*)

Problem 27. Georg has placed 1001 buckets on the floor in one long row. In the first round, Georg puts a marble in each of the 1001 buckets. In the second round, he puts a marble in every second bucket, beginning with the first bucket. He continues like this for 1000 rounds where in round n he puts a marble in every nth bucket, always starting by putting a marble in the first bucket. How many marbles does the last bucket contain at the end?

(*Georg Mohr 2018, problem 18*)

Problem 28. The numbers a, b, c, d, e and f are positive integers below 1000, and they satisfy

$$4a=5b=6c=7d=8e=9f\ .$$

What is the number a?

(*Georg Mohr 2016, problem 19*)



Solution sketches

Problem 1 Answer: 4. By using the digital sum rule for divisibility by 9, we can immediately see that 9393939 and 111111111 are divisible by 9, and that 123456 is not. The number 123456 has digital sum 21, and it is therefore divisible by 3. The number 15 is also divisible by 3, so their product $123456 \cdot 15$ is divisible by $9 = 3^2$. The number $99 \cdot 111$ is divisible by 9, as one of the factors is divisible by 9.

Problem 2 Answer: B. The number 3 does not divide $12445+54421 = 66866$ as it does not divide the digital sum. Using the digital sum rule, it is easy to see that it divides the four other options.

Problem 3 Answer: C. Because x is the product of 9 and some other integer, 3 divides x because 3 divides 9. We can exclude the other possibilities: If Georg thinks of 2, then $x = 27$, which excludes B, D and E. If Georg thinks of 3, then $x = 72$, which excludes A.

Problem 4 Answer: 1123. If a 3-digit number consists of the digits 1,2 and 3, 3 must divide this number, as it divides its digital sum. This means that the number we're looking for must have at least 4 digits. The smallest 4-digit number only consisting of the digits 1, 2 and 3, containing each digit at least once, is 1123, which is not divisible by 3 as its digital sum is not divisible by 3.

Problem 5 Answer: D.

A) Because 4 divides 100, we can check if 4 divides a number just by checking the last two digits. For example, 4 divides 13524, because it divides 24.

B) Because 5 divides a number if it ends in 0 or 5, 5 divides, for example, 12345.

C) Because 6 divides a number if and only if both 2 and 3 divides it, 6 divides, for example, 54132.

D) The alternating digital sum of a number with the digits 1, 2, 3, 4 and 5 can never be divisible by 11, because the digital sum will always be between -9 and 9 because $5+4+3-2-1 = 9$, and it can never be 0 as there are three odd digits. For that reason, 11 does not divide any number consisting of the digits 1, 2, 3, 4 and 5.

E) Because 125 divides 1000, we can check the last three digits of a number to see if it is divisible by 125. For example, 125 divides 24125, as it divides itself.

Problem 6 Answer: D. If $x + y$ is odd, one of the numbers is odd and the other even. If $x \cdot y$ is odd, both numbers are odd. A and D therefore cannot be simultaneously be speaking the truth. This means that B, C and E speak the truth. This means $y = 2x$, showing that y is even, so D is lying. We can see that all others speak the truth if, for example, $x = 3$ and $y = 6$.

Problem 7 Answer: D. If n is divisible by 6, then n must be even. Therefore, both n and m are even, and their sum even as well.Because 2 divides m and 2 divides $n + m$, then $4 = 2 \cdot 2$ divides $m(n + m)$. None of the other options are always divisible by 4: A) $n + m$ is not always divisible by four, for example, if $n = 2$ and $m = 12$. B) $nm - m$, C) $m^2 + n$ and E) $n(m + 1)$ are not always divisible by 4, for example if $n = 2$ and $m = 6$.

Problem 8 Answer: B. Bent receives the presents left from 3, 6 and 9 if he goes up last. Because none of the others' favourite numbers divide either 3 or 9, they must be remaining at the end, while 6 is taken by either Arne or Ella. This means Bent gets two presents if he goes up last. This does not hold for any of the others. If Arne goes up last, he only gets present number 2. If Carl or Ella goes last, they get no presents. If Dora goes last, she gets present number 5.

Problem 9 Answer: D. By looking at the prime factor decomposition $2^9 \cdot 3 \cdot 7 \cdot 11 \cdot 13$ we see that the number $121 = 11^2$ is not a divisor. It's easy to see, using prime factor decomposition, that $6 = 2 \cdot 3$, $14 = 2 \cdot 7$, $39 = 3 \cdot 13$ and $64 = 2^6$ all divide $2^9 \cdot 3 \cdot 7 \cdot 11 \cdot 13$.

Problem 10 Answer: C. Because the gold coins can be distributed in bags that each contain 49 coins, the number of coins is divisible by $49 = 7^2$. Because the gold coins can be distributed in 35 boxes each containing the same amount of coins, the number of coins is divisible by $35 = 5 \cdot 7$. This means that $5 \cdot 7^2 = 245$ divides the number of coins. It's therefore possible that the main has $2450 = 245 \cdot 10$ coins. The other rumours are false, because 245 divides none of the other amounts.



Problem 11 Answer: 32. The prime factor decomposition of 1001 is $7 \cdot 11 \cdot 13$. If the product of four distinct integers is equal to 1001, these must then be 1, 7, 11 and 13. Their sum is 32.

Problem 12 Answer: 450. Because 3 and 5 divide n, then $3 \cdot 5 = 15$ must divide n. Note that of the numbers $1,2,3,4,5,6,7,8,9,10,11,12,13,14,15$, $3,5,6,9,10,12,15$ are removed, that is, there are 8 number left. This pattern repeats for the next 15 numbers and so on. At the end there are $240 = 8 \cdot 30$ numbers left, so from the start we had $n = 15 \cdot 30 = 450$ numbers..

Problem 13 Answer: C. The integers a, b and c must satisfy that

$$a^2 b^3 c^4 = -27 = -3^3.$$

So c^4 must divide 3^3. The number c^4 is a positive integer, and the only positive integers that divide 3^3 are 1, 3, 3^2 and 3^3, and of them, only 1 could be equal to c^4. The only integers c that are solutions of $c^4 = 1$, are $c = 1$ and $c = -1$. These both satisfy the equation when $a = 1$ and $b = -3$. Thus, there are two possibilities for c.

Problem 14 Answer: 11. The divisors of p^{10} are exactly $1, p, p^2, p^3, \ldots, p^{10}$.

Problem 15 Answer: 1. As the last digit of a product only depends on the last digit of each factor, the last digit of 74851^3 is equal to the last digit of $1^3 = 1$, that is, 1.

Problem 16 Answer: 9. Because $9^2 = 81$ has 1 as its last digit, then $9^3 = 9 \cdot 9^2$ must have the same last digit as $9 \cdot 1 = 9$. Similarly, $9^4 = 9 \cdot 9^3$ must have the same last digit as $9 \cdot 9 = 81$, that is, 1, and we go like that. This shows that the last digit of 9^n is 1 if n is even, and 9 if n is odd. Thus, the last digit of 9^{2013} is 9.

Problem 17 Answer: E.

A) The last digit of the number $5+15+25+35+45$ is the last digit of $5+5+5+5+5 = 25$, which is 5.

B) The last digit of $5 \cdot 15 \cdot 25 \cdot 35 \cdot 45$ is the last digit of 5^5. When you multiply 5 by 5, the last digit is again 5, so the last digit is 5.

C) The last digit 19^2 is the last digit of $9^2 = 81$, which is 1. The last digit of 16^2 is the last digit of $6^3 = 36$, which is 6. Thus, the last digit of the difference $19^2 - 16^2$ is 5.

D) The last digit of the number $\frac{20 \cdot 35 \cdot 9}{4 \cdot 15} = \frac{5 \cdot 35 \cdot 9}{3 \cdot 5} = 5 \cdot 7 \cdot 3$ is 5 as it is divisible by 5, but not by 10.

E) The last digit of 25^{25} is the same as the last digit of 5^{25}, and as before we know that the last digit of a power of 5 is 5. Similarly, the last digit of 25^5 is also 5. Thus the last digit of the difference $25^{25} - 25^5$ is equal to the last digit of lig $5 - 5 = 0$, which is 0.

Problem 18 Answer: 6. The number

$$2000^{1000} = 2^{1000} \cdot 1000^{1000} = 2^{1000} \cdot 10^{3000},$$

ends in 3000 zeroes. The digit just before the 3000 zeroes must be the last digit of 2^{1000}. Looking at the powers of $2, 2, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9 \ldots$ we note that their last digits are $2,4,8,6,2,4,8,6,2,\ldots$, and this must continue periodically because the last digit of a product only depends on the last digits of the factors. Because 1000 is divisible by 4, the last digit of 2^{1000} must be 6.

Problem 19 Answer: 4. The last digit of a product only depends on the last digits of each factor. To find the possible last digits of n^{16} for $n = 1,2,\ldots 100$, we therefore only need to check $n = 0,1,2,\ldots,9$. We use that $n^{16} = (((n^2)^2)^2)^2$. To find the possible last digits of n^2 we first find the last digits of $0^2, 1^1, \ldots, 9^2$, and conclude that the possibilities are $0,1,4,5,6,9$. We can now find the possible last digits of $n^4 = (n^2)^2$ by looking at the last digits of $0^2, 1^2, 4^2, 5^2, 6^2, 9^2$, which are $0,1,5,6$. Repeating this, we can now find the possible last digits of $n^4 = (n^2)^2$ by looking at the last digits of $0^2, 1^2, 4^2, 5^2, 6^2, 9^2$, which are $0,1,5,6$. Repeating it once again to find the possible last digits of $n^{16} = (n^8)^2$, we yet again obtain $0,1,5,6$.

Problem 20 Answer: E. If $4n+1$ divides

$$12n + 10 = 3(4n+1) + 7,$$

then $4n+1$ also divides 7. There is no positive integer n such that $4n+1$ divides 7: When $n = 1$, $4n+1 = 5$. When $n \geq 2$, $4n+1 \geq 9$ which cannot divide 7. This means the answer is E).



Problem 21 Answer: B. A number is divisible by 18 if and only if it is divisible by both 9 and 2, as 9 and 2 share no divisors. You can check if 9 divides a number by checking if it divides its digital sum.

A) 540: Both 9 and 2 are divisors, then 18 is as well.

B) $1818+7216$: 9 divides 1818 but not 7216, and then doesn't divide their sum. Thus neither does 18.

C) $36 \cdot 4226$: Because 18 divides 36, it also divides a multiple of 36.

D) $2007 \cdot 2008 \cdot 2009$. Because 9 divides 2007 and 2 divides 2008, 18 divides the product of two of the three factors, and thus the product of all three factors.

E) $9 \cdot 888 - 8 \cdot 999$: Because 9 divides 9 and 2 divides 888, 18 divides their product $9 \cdot 888$. Similarly, we see that 18 divides$8 \cdot 999$. Because 18 divides each number, it also divides their difference.

Problem 22 Answer: 3. The number $2n+1$ divides

$$8n+69 = 4(2n+1)+65,$$

if and only if it divides 65. The divisors of $65 = 5 \cdot 13$ are 1, 5, 13 and 65. The number $2n+1$ is greater than one, so must be equal to 5, 13 or 65. Thus, n is equal to 2, 6 or 32.

Problem 23 Answer: A). Because $n+m = n^2$, n must divide $m = n^2-n$, because n divides both n^2 and n and therefore also their difference.

Because $n = 3$ and $m = 6$ satisfy $n+m = n^2$, we cannot deduce any of B, C, D or E.

Problem 24 Answer: 15. Peter eats n chocolate frogs, and the problem states that $n > 1$. Because Peter eats $\frac{1}{4}$ frogs, he gets $4n$ frogs. Let m be the number of people sharing the chocolate frogs. We know that Peter has more than one friend, so m is at least 3. Because each person gets $4n$ frogs, we know that

$$4n \cdot m = 100 \quad \Leftrightarrow n \cdot m = 25 = 5^2.$$

We know that n and m are two positive integers larger than 1 with the product 5^2, and thus the only possibility is $n = m = 5$. Peter gets $4 \cdot 5 = 20$ frogs, and as he eats 5, he has 15 remaining.

Problem 25 Answer: 40. If a person draws two notes where $6 = 2 \cdot 3$ divides the result, then 3 must divide at least one of the two numbers. There are only 20 notes in each box where 3 is a divisor, so at most 40 people can draw a note divisible by 3. It is a possibility that 40 sodas are needed: In each box there are 20 numbers divisible by 3, and 20 even numbers in addition to these. If the 20 numbers divisible by 3 are combined with the 20 even numbers not divisible by 3 from the other box, all of these combinations give sodas, equivalently if swapping the boxes.

Problem 26 Answer: 15. First, we rewrite the equation:

$$\begin{aligned} 5x^5+3x^3+x &= 3604065+4x^4+2x^2 \quad \Leftrightarrow \\ 3604065 &= x(5x^4+3x^2+1-4x^3-2x). \end{aligned}$$

We know that one of the integers 14, 15, 16, 20 and 25 is a solution to the equation. The number x cannot be even, as that would make the right side even and the left side odd. This excludes 14, 16 and 20 as solutions. x can also not be 25, as that would make 25 divide the right side by not the left side: 25 does not divide 3604065 because 25 divides a number if and only if it divides its last two digits. This means that $x = 15$ is a solution.

Problem 27 Answer: 16. Enumerate the buckets $0, 1, 2, 3, \ldots, 1000$. In round k Georg puts marbles in every bucket with a number that is divisible by k. For example, in round 3 he puts marbles in buckets number $0, 3, 6, \ldots, 999$. The last bucket has number $1000 = 2^3 \cdot 5^3$. At the end, it will have a marble for each divisor of 1000. There are 16 of those, which are all the combinations of $1, 2, 2^2, 2^3$ multiplied by $1, 5, 5^2, 5^3$:

	1	2	2^2	2^3
1	1	2	2^2	2^3
5	5	$2 \cdot 5$	$2^2 \cdot 5$	$2^3 \cdot 5$
5^2	5^2	$2 \cdot 5^2$	$2^2 \cdot 5^2$	$2^3 \cdot 5^2$
5^3	5^3	$2 \cdot 5^3$	$2^2 \cdot 5^3$	$2^3 \cdot 5^3$



Problem 28 Answer: 630. Looking at

$$4a = 5b = 6c = 7d = 8e = 9f \ ,$$

we observe that:

- $4a = 5b$, that is, 5 divides $4a$ and thus also divides a, because 5 is a prime number not dividing 4.
- $2a = 3c$, that is, 3 divides $2a$ and thus also divides a because 3 is a prime number not dividing 2.
- $4a = 7d$, that is, 7 divides $4a$ and thus also divides a because 7 is a prime number not dividing 4.
- $a = 2e$, that is, 2 divides a.
- $4a = 3^2 f$, that is, 3^2 divides $4a$ and thus also divides a because 7 is a prime number not dividing 4.

Because all of 2, 3, 3^2, 5 and 7 must divide a, then $2 \cdot 3^2 \cdot 5 \cdot 7 = 630$ must divide a because $2 \cdot 3^2 \cdot 5 \cdot 7$ is the smallest number divisible by all of 2, 3, 3^2, 5 and 7. Because a is at most 1000, we know that $a = 630$. Note that the equations are true for $a = 2 \cdot 3^2 \cdot 5 \cdot 7 = 630$, $b = 2^3 \cdot 3^2 \cdot 7 = 504$, $c = 2^2 \cdot 3 \cdot 5 \cdot 7 = 420$, $d = 2^3 \cdot 3^2 \cdot 5 = 360$, $e = 3^2 \cdot 5 \cdot 7 = 315$, $f = 2^3 \cdot 5 \cdot 7 = 280$.

Translated by Andreas Munch Mikkelsen