The 36th Nordic Mathematical Contest

Monday, 4 April 2022

English version

Time allowed: 4 hours. Each problem is worth 7 points.
Only writing and drawing tools are allowed.

Problem 1

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)f(1-x)) = f(x) \qquad \text{and} \qquad f(f(x)) = 1 - f(x),$$

for all real x.

Problem 2

In Wonderland, the towns are connected by roads, and whenever there is a direct road between two towns there is also a route between these two towns that does not use that road. (There is at most one direct road between any two towns.) The Queen of Hearts ordered the Spades to provide a list of all "even" subsystems of the system of roads, that is, systems formed by subsets of the set of roads, where each town is connected to an even number of roads (possibly none). For each such subsystem they should list its roads. If there are totally n roads in Wonderland and x subsystems on the Spades' list, what is the number of roads on their list when each road is counted as many times as it is listed?

Problem 3

Anton and Britta play a game with the set $M = \{1, 2, 3, \ldots, n-1\}$ where $n \geq 5$ is an odd integer. In each step Anton removes a number from M and puts it in his set A, and Britta removes a number from M and puts it in her set B (both A and B are empty to begin with). When M is empty, Anton picks two distinct numbers x_1, x_2 from A and shows them to Britta. Britta then picks two distinct numbers y_1, y_2 from B. Britta wins if

$$(x_1 x_2 (x_1 - y_1)(x_2 - y_2))^{\frac{n-1}{2}} \equiv 1 \mod n,$$

otherwise Anton wins. Find all n for which Britta has a winning strategy.

Problem 4

Let ABC be an acute-angled triangle with circumscribed circle k and centre of the circumscribed circle O. A line through O intersects the sides AB and AC at D and E, respectively. Denote by B' and C' the reflections of B and C over O, respectively. Prove that the circumscribed circles of ODC' and OEB' concur on k.