The 31st Nordic Mathematical Contest

Monday, 3 April 2017

Solutions

Problem 1 Let n be a positive integer. Show that there exist positive integers a and b such that:

$$\frac{a^2+a+1}{b^2+b+1} = n^2+n+1.$$

Solution 1 Let $P(x) = x^2+x+1$. We have $P(n)P(n+1) = (n^2+n+1)(n^2+3n+3) = n^4+4n^3+7n^2+6n+3$. Also, $P((n+1)^2) = n^4+4n^3+7n^2+6n+3$. By choosing $a = (n+1)^2$ and $b = n+1$ we get $P(a)/P(b) = P(n)$ as desired.

Problem 2 Let a, b, α, β be real numbers such that $0 \le a, b \le 1$, and $0 \le \alpha, \beta \le \frac{\pi}{2}$. Show that if

$$ab\cos(\alpha-\beta) \le \sqrt{(1-a^2)(1-b^2)},$$

then

$$a\cos\alpha + b\sin\beta \le 1 + ab\sin(\beta-\alpha).$$

Solution 2 The condition can be rewritten as

$$ab\cos(\alpha-\beta) = ab\cos\alpha\cos\beta + ab\sin\alpha\sin\beta \le \sqrt{(1-a^2)(1-b^2)}.$$

Set $x = a\cos\alpha,\ y = b\sin\beta,\ z = b\cos\beta,\ t = a\sin\alpha$. We can now rewrite the condition as

$$xz + yt \le \sqrt{(1-x^2-t^2)(1-y^2-z^2)},$$

whereas the inequality we need to prove now looks like

$$x + y \le 1 + xy - zt.$$

Since $x, y, z, t \ge 0$, and $1 + xy - zt = 1 + ab\sin(\beta-\alpha) \ge 0$, we can square both sides of both inequalities, and get equivalent ones. After a couple of cancelations the condition yields

$$2xyzt \le 1 - x^2 - y^2 - z^2 - t^2 + x^2y^2 + z^2t^2,$$

so that

$$x^2+y^2+z^2+t^2 \le (xy-zt)^2 + 1,$$

which is equivalent to

$$x^2 + y^2 + z^2 + t^2 + 2xy - 2zt \leq (1 + xy - zt)^2,$$

or

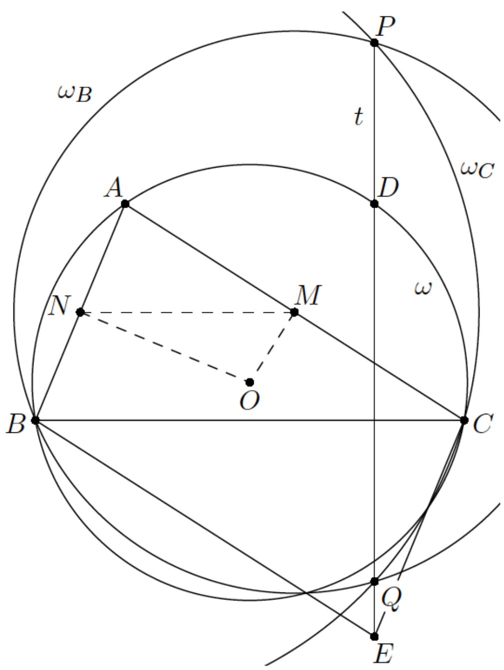
$$(x + y)^2 + (z - t)^2 \leq (1 + xy - zt)^2.$$

Since $(x + y)^2 \leq (x + y)^2 + (z - t)^2$, the desired inequality follows.

Problem 3 Let M and N be the midpoints of the sides AC and AB, respectively, of an acute triangle ABC, $AB \neq AC$. Let ω_B be the circle centered at M passing through B, and let ω_C be the circle centered at N passing through C. Let the point D be such that $ABCD$ is an isosceles trapezoid with AD parallel to BC. Assume that ω_B and ω_C intersect in two distinct points P and Q. Show that D lies on the line PQ.

Solution 3 Let E be such that $ABEC$ is a parallelogram with $AB \parallel CE$ and $AC \parallel BE$, and let ω be the circumscribed circle of $\triangle ABC$ with centre O.

It is known that the radical axis of two circles is perpendicular to the line connecting the two centres. Since $BE \perp MO$ and $CE \perp NO$, this means that BE and CE are the radical axes of ω and ω_B, and of ω and ω_C, respectively, so E is the radical centre of ω, ω_B, and ω_C.



Now as $BE = AC = BD$ and $CE = AB = CD$ we find that BC is the perpendicular bisector of DE. Most importantly we have $DE \perp BC$. Denote by t the radical axis of ω_B and ω_C, i.e. $t = PQ$. Then since $t \perp MN$ we find that t and DE are parallel. Therefore since E lies on t we get that D also lies on t.

Alternative solution Reflect B across M to a point B' forming a parallelogram $ABCB'$. Then B' lies on ω_B diagonally opposite B, and since $AB' \parallel BC$ it lies on AD. Similarly reflect C across N to a point C', which satisfies analogous properties. Note that $CB' = AB = CD$, so we find that triangle CDB' and similarly triangle BDC' are isosceles.

Let B'' and C'' be the orthogonal projections of B and C onto AD. Since BB' is a diameter of ω_B we get that B'' lies on ω_B, and similarly C'' lies on ω_C. Moreover BB'' is an altitude of the isosceles triangle BDC' with $BD = BC'$, hence it coincides with the median from B, so B'' is in fact the midpoint of DC'. Similarly C'' is the midpoint of DB'. From this we get

$$2 = \frac{DC'}{DB''} = \frac{DB'}{DC''}$$

which rearranges as $DC' \cdot DC'' = DB' \cdot DB''$. This means that D has same the power with respect to ω_B and ω_C, hence it lies on their radical axis PQ.

Problem 4 Find all integers n and m, $n > m > 2$, and such that a regular n-sided polygon can be inscribed in a regular m-sided polygon so that all the vertices of the n-gon lie on the sides of the m-gon.

Solution 4 It works only for $n = 2m$, and for $m = 3$ and $n = 4$.

To begin with let's see why it works for $n = 2m$. For a $2m$-gon we can choose two points on each side, symmetrically, so that the distance between the two of them is equal to the distance between two close points on adjacent sides.

For $n = 4$ and $m = 3$ we need to inscribe a square in an equilateral triangle, by choosing two vertices of the square symmetrically on one of the sides of the triangle. It is easy to calculate the side length of the square so that its remaining two vertices lie on the remaining two sides of the triangle.

We need to show that it cannot be done in any other way. One side of the m-gon can contain at most two of the vertices of the n-gon, so that $n \leq 2m$. For $n \geq m+2$ at least two of the sides of the m-gon must contain two of the vertices of the n-gon each. By symmetry the midpoints, and thus the perpendicular bisectors, of such a side of the m-gon and of the side of the n-gon it contains must coincide. If two such sides of the m-gon are not opposite to each other the corresponding perpendicular bisectors will intersect, and we can deduce that the centres of the circumscribed circles of the m-gon and of the n-gon must coincide. If two such sides are opposite, then the centres of the circumscribed circles will coincide with the midpoint of the segment between the midpoints of the sides, and thus the two circumscribed circles will once again have the same centre.

Denote the radii of the two circumscribed circles by R and r, where $R > r$. The smaller circle intersects the sides of the m-gon at $2m$ points, among which are the possible vertices for the n-gon. Denote these points by $P_1, P_2, \ldots, P_{2m}$ clockwise, where P_1 and P_2

are vertices of the n-gon. If the side length of the n-gon is s, we now have $|P_1P_2| = s$, and thus $|P_3P_4| = s$. If only one of the points P_3 and P_4 is a vertex in the n-gon, it would have to be P_4, but the distance between P_2 and P_4 is greater than s, which means P_2P_4 cannot be a side. We can now deduce that both P_3 and P_4 are vertices of the n-gon, and by symmetry all the $2m$ points of intersection will be vertices of the n-gon, so that $n = 2m$.

We now need to handle the case $n = m + 1 > 4$. Denote the vertices of the m-gon by $Q_1, Q_2, \ldots, Q_m$ clockwise. Now only one of the sides of the m-gon contains two of the vertices of the n-gon, let this side be Q_mQ_1. Denote the vertices of the $(m+1)$-gon by $P_1, P_2, \ldots, P_{m+1}$ clockwise, where P_1 and P_{m+1} lie on the side Q_mQ_1 of the m-gon. Let $\alpha = \pi - 2\pi/(m+1)$, and $\beta = \pi - 2\pi/m$ be the angles of the n-gon and the m-gon, respectively. We now get a number of triangles $P_1Q_1P_2$, $P_2Q_2P_3$,… $P_mQ_mP_{m+1}$. We begin by establishing a connection between the sizes of the angles. To begin with we have $\angle Q_1P_1P_2 = \pi - \alpha$, and $\angle P_1Q_1P_2 = \beta$, so that $\angle P_1P_2Q_1 = \alpha - \beta$. We now proceed to get $\angle Q_2P_2P_3 = \pi - \alpha - (\alpha - \beta)$, and $\angle P_2Q_2P_3 = \beta$, which gives $\angle P_2P_3Q_2 = 2(\alpha - \beta)$, and so on. If now $\gamma = \alpha - \beta = 2\pi/m(m+1)$, we have $\angle P_kP_{k+1}Q_k = k\gamma$, and $\angle Q_kP_kP_{k+1} = (m+1-k)\gamma$ for $k = 1, 2, \ldots, m$.

Since s is the side length of the n-gon, i.e. $s = |P_1P_2| = |P_1P_2| = \cdots = |P_{m+1}P_1|$, according to the law of sines we get

$$\frac{s}{\sin\beta} = \frac{|P_kQ_k|}{\sin(P_kP_{k+1}Q_k)} = \frac{|Q_kP_{k+1}|}{\sin(Q_kP_kP_{k+1})},$$

i.e.

$$\frac{s}{\sin\beta} = \frac{|P_kQ_k|}{\sin(k\gamma)} = \frac{|Q_kP_{k+1}|}{\sin((m+1-k)\gamma)}.$$

Since $|Q_kQ_{k+1}| = |Q_kP_{k+1}| + |P_{k+1}Q_{k+1}|$, we get

$$\sin((k+1)\gamma) + \sin((m+1-k)\gamma) = \frac{\sigma \sin\beta}{s}$$

where $\sigma = |Q_1Q_2| = |Q_2Q_3| = \cdots = |Q_mQ_1|$ is the side length of the m-gon.

Using the above for $k = 1$ and $k = 2$, we get

$$\sin 2\gamma + \sin m\gamma = \sin 3\gamma + \sin(m-1)\gamma.$$

For $m \geq 4$ the angles 3γ and $(m-1)\gamma$ are both in the interval between 2γ and $m\gamma$, which means we can't have equality as the sine function is concave (convex from above) in the interval $[0, \pi/2]$. We therefore deduce that it is impossible to inscribe an $(m+1)$-gon in an m-gon for $m \geq 4$.