24th Nordic Mathematical Contest, 13th of April, 2010

Solutions of the problems

1. *A function $f\colon \mathbb{Z}_+ \to \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of positive integers, is non-decreasing and satisfies $f(mn) = f(m)f(n)$ for all relatively prime positive integers m and n. Prove that $f(8)f(13) \geq (f(10))^2$.*

Solution: Since f is non-decreasing, $f(91) \geq f(90)$, which (by factorization into relatively prime factors) implies $f(13)f(7) \geq f(9)f(10)$. Also $f(72) \geq f(70)$, and therefore $f(8)f(9) \geq f(7)f(10)$. Since all values of f are positive, we get

$$f(8)f(9) \cdot f(13)f(7) \geq f(7)f(10) \cdot f(9)f(10),$$

and dividing both sides by $f(7)f(9) > 0$,

$$f(8)f(13) \geq f(10)f(10) = (f(10))^2. \quad \square$$

Remark: More generally, it can be shown that every multiplicative non-decreasing function $f\colon \mathbb{Z}_+ \to \mathbb{Z}_+$ is a power function, i.e., there is $k \in \mathbb{N}$ such that for every $n \in \mathbb{Z}_+$, we have $f(n) = n^k$. Once this is known, the solution is of course trivial. This result is a direct corollary of a theorem by Erdős, published in *On the distribution function of additive functions*, **Ann. of Math.** (2) 47 (1946), 1–20, Theorem XI.

2. *Three circles Γ_A, Γ_B and Γ_C share a common point of intersection O. The other common point of Γ_A and Γ_B is C, that of Γ_A and Γ_C is B, and that of Γ_C and Γ_B is A. The line AO intersects the circle Γ_A in the point $X \neq O$. Similarly, the line BO intersects the circle Γ_B in the point $Y \neq O$, and the line CO intersects the circle Γ_C in the point $Z \neq O$. Show that*

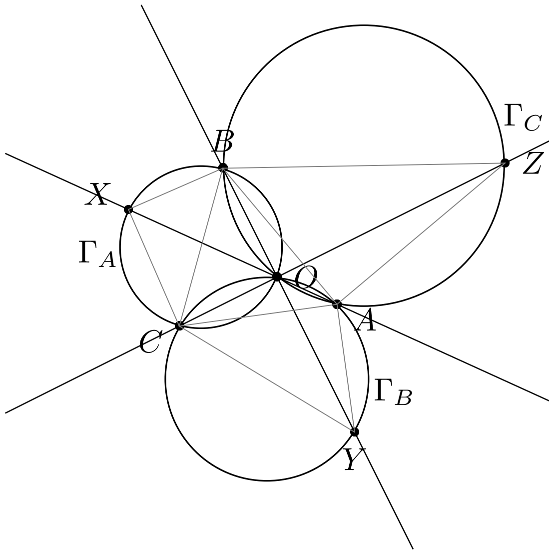
$$\frac{|AY|\,|BZ|\,|CX|}{|AZ|\,|BX|\,|CY|} = 1.$$

Solution: Observe $\measuredangle AOY = \measuredangle BOX = \alpha$ (vertical angles). By looking on peripheral angles we further get

$$\alpha = \measuredangle BCX = \measuredangle BOX = \measuredangle AOY = \measuredangle ACY.$$

In the same manner we get

$$\beta = \measuredangle BAZ = \measuredangle BOZ = \measuredangle COY = \measuredangle CAY,$$
$$\gamma = \measuredangle ABZ = \measuredangle AOZ = \measuredangle COX = \measuredangle CBX.$$



We have $\alpha + \beta + \gamma = \measuredangle AOY + \measuredangle YOC + \measuredangle COX = 180°$ and hence all triangels CAY, CXB and ZAB have angles α, β and γ and hence they are similar. This gives $\frac{|AY|}{|CY|} = \frac{|AB|}{|BZ|}$ and $\frac{|CX|}{|BX|} = \frac{|AZ|}{|AB|}$, i.e.

$$\frac{|AY|\,|BZ|\,|CX|}{|AZ|\,|BX|\,|CY|} = \frac{|AB|\,|AZ|\,|BZ|}{|BZ|\,|AB|\,|AZ|} = 1. \quad \square$$

3. *Laura has 2010 lamps connected with 2010 buttons in front of her. For each button, she wants to know the corresponding lamp. In order to do this, she observes which lamps are lit when Richard presses a selection of buttons. Richard always presses the buttons simultaneously, so the lamps are lit simultaneously, too.*

a) *If Richard chooses the buttons to be pressed, what is the maximum number of different combinations of buttons he can press until Laura can assign the buttons to the lamps correctly?*
b) *Supposing that Laura will choose the combinations of buttons to be pressed, what is the minimum number of attempts she has to do until she is able to associate the buttons with the lamps in a correct way?*

Solution: a) Let us say that two lamps are separated if one of the lamps is turned on while the other lamp remains off. Laura can find out which lamps belong to the buttons if every two lamps are separated. Let Richard choose two arbitrary lamps. To begin with, he turns both lamps on and then varies all the other lamps in all possible ways. There are 2^{2008} different combinations for the remaining $2010 - 2 = 2008$ lamps. Then Richard turns the two chosen lamps off. Also, at this time there are 2^{2008} combinations for the remaining lamps. Consequently, for the 2^{2009} combinations in all, it is not possible to separate the two lamps of the first pair. However, we cannot avoid the separation if we add one more combination. Indeed, for every pair of lamps, we see that if we turn on a combination of lamps $2^{2009} + 1$ times, there must be at least one setup where exactly one of the lamps is turned on and the other is turned off. Thus, the answer is $2^{2009} + 1$.

b) For every new step with a combination of lamps turned on, we get a partition of the set of lamps into smaller and smaller subsets where elements belonging to the same subset cannot be separated. In each step every subset is either unchanged or divided into two smaller parts, i.e. the total number of subsets after k steps will be at most 2^k. We are finished when the number of subsets is equal to 2010, so the answer is at least $\lceil \log_2 2010 \rceil = 11$. But it is easy to see that Laura certainly can choose buttons in every step in such a way that there are at most 2^{11-k} lamps in every part of the partition after k steps. Thus, the answer is 11.

Remark: More formally, let B be the set of 2010 buttons and L be the set of lamps. The wording of the problem should be interpreted so that the connections between buttons and lamps form a bijection $f\colon B \to L$ and the task is to determine that f. During the process, distinct combinations $B_k \subset B$ of buttons are pressed, and lamps $f[B_k]$ are lit, $k = 1, 2, \dots, s$. Laura is not finished, unless for every bijection $g\colon B \to L$ with the property that $g[B_k] = f[B_k]$ holds for every $k = 1, \dots, s$, we have that $g = f$. Let us elaborate this a bit: The sets $f[B_1], \dots, f[B_k]$ generate a partition Π_k of L. Then lamps x and y are separated iff they belong to different parts. Furthermore, f is not uniquely determined, unless all lamps are separated. The ideas written above may now be carried over in this more formal setting.

4. *A positive integer is called simple if its ordinary decimal representation consists entirely of zeroes and ones. Find the least positive integer k such that each positive integer n can be written as $n = a_1 \pm a_2 \pm a_3 \pm \dots \pm a_k$ where $a_1, \dots, a_k$ are simple.*

Solution: First we observe that if a positive integer n has a representation in terms of l simple numbers, then it has also a representation in terms of $l + 1$ simple numbers. Indeed, suppose $n = a_1 \pm a_2 \pm a_3 \pm \dots \pm a_l$ is such a representation. Let 10^r, $r \in \mathbb{N}$, be bigger than any of the numbers $a_1, \dots, a_l$, and put $a_1' = 10^r + a_1$. Then obviously a_1' is also simple and $n = a_1' \pm a_2 \pm a_3 \pm \dots \pm a_l - 10^r$ is a required representation.

Let n be an arbitrary positive integer. We write $n = a_1 + a_2 + \dots + a_9$, where a_j has 1's on the places where n has digits greater or equal to j and 0's on the other places. Then $n = a_1 + a_2 + \dots + a_9$, each a_i is either simple or zero, so n has been represented in terms of at most 9 simple integers. According to the preceding paragraph, this can be made exactly 9, and we get $k \leq 9$.

On the other hand, consider $n = 10203040506070809$. Suppose $n = a_1 + a_2 + \dots + a_j - a_{j+1} - a_{j+2} - \dots - a_k$ where $a_1, \dots a_k$ are simple, $k < 9$. Then all digits of $b_1 = a_1 + \dots + a_j$ are not greater than j and all digits of $b_2 = a_{j+1} + \dots + a_k$ are not greater than $k - j$. We have $n + b_2 = b_1$. We perform column addition of n and b_2 and consider digit $j + 1$ in the number n. There will be no summand coming from lower decimal places, since the sum there is less that $10\dots0 + 88\dots8 = 98\dots8$. So we get the sum of $j + 1$ and the corresponding digit in b_2, the resulting digit should be less than $j + 1$ thus in b_2 we have at least $9 - j \leq k - j$, implying $k \geq 9$.

Hence, we have proved that $k = 9$.

Alternative solutions: The number n above is well chosen, but there are infinitely many other examples that are good for proving the lower bound $k \geq 9$. Let me sketch the main idea: Suppose n can be represented by simple numbers, first adding p simple numbers and the subtracting further m numbers. Suppose $p+m \leq 9$. Write the decimal representations: $n = \sum_{i=0}^{r} a_i 10^i$, let the sum of positively occurring simple numbers be $\sum_{i=0}^{r} p_i 10^i$ and let the sum of negatively occurring simple numbers be $\sum_{i=0}^{r} m_i 10^i$. Then $a_0 \equiv p_0 - m_0 \pmod{10}$ and for $s \in \mathbb{N}$, $0 < s \leq r$, we have either $a_s \equiv p_s - m_s \pmod{10}$ or $a_s \equiv p_s - m_s - 1 \pmod{10}$. A case analysis shows for, say, $n = 123456789$ or $n = 987654321$, that $p + m = 9$.