The 22nd Nordic Mathematical Contest

31 March 2008
Solutions

Time allowed is 4 hours. Each problem is worth 5 points. The only permitted aids are writing and drawing materials.

Problem 1

Determine all real numbers A, B and C such that there exists a real function f that satisfies

$$f(x + f(y)) = Ax + By + C$$

for all real x and y.

Solution. Let A, B and C be real numbers and f a function such that $f(x + f(y)) = Ax + By + C$ for all x and y.

Let z be a real number and set $x = z - f(0)$ and $y = 0$. Then

$$f(z) = f(z - f(0) + f(0)) = A(z - f(0)) + B \cdot 0 + C = Az - Af(0) + C,$$

so there are numbers a and b such that $f(z) = az + b$ for all z. Now $f(x + f(y)) = ax + a^2y + (a + 1)b$, and $(A, B, C) = (a, a^2, (a + 1)b)$, where a and b are arbitrary real numbers, that is, $(A, B, C) = (a, a^2, c)$, where $a \neq -1$ and c are arbitrary, or $(A, B, C) = (-1, 1, 0)$.

Problem 2

Assume that $n \geq 3$ people with different names sit around a round table. We call any unordered pair of them, say M and N, *dominating*, if

(i) M and N do not sit on adjacent seats, and

(ii) on one (or both) of the arcs connecting M and N along the table edge, all people have names that come alphabetically after the names of M and N.

Determine the minimal number of dominating pairs.

Solution. We will show by induction that the number of dominating pairs (hence also the minimal number of dominating pairs) is $n-3$ for $n \geq 3$.

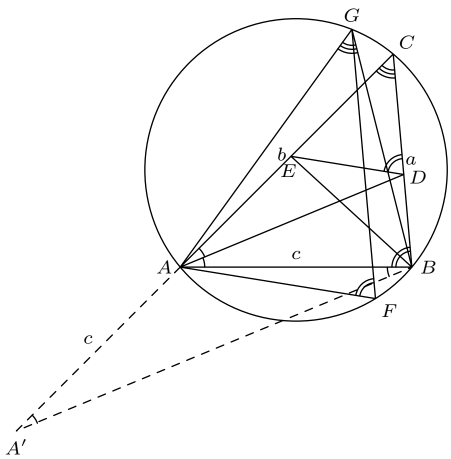
If $n = 3$, all pairs of people sit on adjacent seats, so there are no dominating pairs. Assume that the number of dominating pairs is $n-3$ for some $n \geq 3$. If there are $n+1$ people around the table, let the person whose name is alphabetically last leave the table. The two people sitting next to that person, who formed a dominating pair, no longer do. On the other hand, any other dominating pair remains a dominating pair in the new configuration of n people, and any dominating pair in the new configuration was also a dominating pair in the old. The number of dominating pairs in the new configuration is $n-3$, so the number in the old was $(n+1)-3$.

Problem 3

Let ABC be a triangle and let D and E be points on BC and CA, respectively, such that AD and BE are angle bisectors of ABC. Let F and G be points on the circumcircle of ABC such that AF and DE are parallel and FG and BC are parallel. Show that

$$\frac{AG}{BG} = \frac{AB+AC}{AB+BC}.$$

Solution. Let $AB = c$, $BC = a$ and $CA = b$. Then it follows from the angle bisector theorem that $CD = ab/(b+c)$.



(The angle bisector theorem can be proved by letting A' be the intersection point of the line CA and the line through B parallel with the bisector AD (dashed lines). Then the angles BAD, ABA', CAD and $CA'B$ are equal, so the triangle $A'AB$ is isosceles, and the equality follows from the similarity of the triangles ACD and $A'CB$.)

Similarly, $CE = ab/(a+c)$, so $CE/CD = (b+c)/(a+c)$. The angles ABG, AFG and EDC are equal, and so are AGB and ACB, and consequently, the triangles CED and GAB are similar. The conclusion follows.

(If, by adding an additional assumption, we make $ABCG$ a convex quadrilateral, we can use Ptolemy's theorem to get the more interesting result that $GA = GB + GC$.)

Problem 4

The difference between the cubes of two consecutive positive integers is a square n^2, where n is a positive integer. Show that n is the sum of two squares.

Solution. Assume that $(m + 1)^3 - m^3 = n^2$. Rearranging we get $3(2m + 1)^2 = (2n + 1)(2n - 1)$. Since $2n + 1$ and $2n - 1$ are relatively prime (if they had a common divisor, it would have divided the difference, which is 2, but they are both odd), one of them is a square (of an odd integer, since it is odd) and the other divided by 3 is a square.

An odd number squared minus 1 is divisible by 4 since $(2t + 1)^2 - 1 = 4(t^2 + t)$. From the first equation we see that n is odd, say $n = 2k + 1$. Then $2n + 1 = 4k + 3$, so the square must be $2n - 1$, say $2n - 1 = (2t + 1)^2$. Rearrangement yields $n = t^2 + (t + 1)^2$.

An example: $8^3 - 7^3 = (2^2 + 3^2)^2$.