47^{th} International Mathematical Olympiad

Slovenia 2006

Problems with Solutions

Contents

Problems

Problem 1. Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Problem 2. Let P be a regular 2006-gon. A diagonal of P is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.

Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Problem 3. Determine the least real number M such that the inequality

$$\left| ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2) \right| \leq M\left(a^2+b^2+c^2\right)^2$$

holds for all real numbers a, b and c.

Problem 4. Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Problem 5. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Problem 6. Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

Solutions

Problem 1.

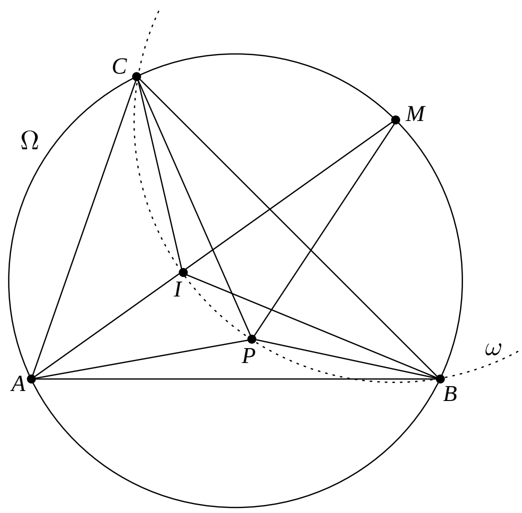
Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Solution. Let $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$. Since $\angle PBA + \angle PCA + \angle PBC + \angle PCB = \beta + \gamma$, the condition from the problem statement is equivalent to $\angle PBC + \angle PCB = (\beta + \gamma)/2$, i. e. $\angle BPC = 90^\circ + \alpha/2$.

On the other hand $\angle BIC = 180^\circ - (\beta + \gamma)/2 = 90^\circ + \alpha/2$. Hence $\angle BPC = \angle BIC$, and since P and I are on the same side of BC, the points B, C, I and P are concyclic. In other words, P lies on the circumcircle ω of triangle BCI.



Let Ω be the circumcircle of triangle ABC. It is a well-known fact that the centre of ω is the midpoint M of the arc BC of Ω. This is also the point where the angle bisector AI intersects Ω.

From triangle APM we have

$$AP + PM \geq AM = AI + IM = AI + PM.$$

Therefore $AP \geq AI$. Equality holds if and only if P lies on the line segment AI, which occurs if and only if $P = I$.

Problem 2.

Let P be a regular 2006-gon. A diagonal of P is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.

Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Solution 1. Call an isosceles triangle *good* if it has two odd sides. Suppose we are given a dissection as in the problem statement. A triangle in the dissection which is good and isosceles will be called *iso-good* for brevity.

Lemma. Let AB be one of dissecting diagonals and let $\mathcal{L}$ be the shorter part of the boundary of the 2006-gon with endpoints A, B. Suppose that $\mathcal{L}$ consists of n segments. Then the number of iso-good triangles with vertices on $\mathcal{L}$ does not exceed $n/2$.

Proof. This is obvious for $n = 2$. Take n with $2 < n \leq 1003$ and assume the claim to be true for every $\mathcal{L}$ of length less than n. Let now $\mathcal{L}$ (endpoints A, B) consist of n segments. Let PQ be the longest diagonal which is a side of an iso-good triangle PQS with all vertices on $\mathcal{L}$ (if there is no such triangle, there is nothing to prove). Every triangle whose vertices lie on $\mathcal{L}$ is obtuse or right-angled; thus S is the summit of PQS. We may assume that the five points A, P, S, Q, B lie on $\mathcal{L}$ in this order and partition $\mathcal{L}$ into four pieces $\mathcal{L}_{AP}$, $\mathcal{L}_{PS}$, $\mathcal{L}_{SQ}$, $\mathcal{L}_{QB}$ (the outer ones possibly reducing to a point).

By the definition of PQ, an iso-good triangle cannot have vertices on both $\mathcal{L}_{AP}$ and $\mathcal{L}_{QB}$. Therefore every iso-good triangle within $\mathcal{L}$ has all its vertices on just one of the four pieces. Applying to each of these pieces the induction hypothesis and adding the four inequalities we get that the number of iso-good triangles within $\mathcal{L}$ other than PQS does not exceed $n/2$. And since each of $\mathcal{L}_{PS}$, $\mathcal{L}_{SQ}$ consists of an odd number of sides, the inequalities for these two pieces are actually strict, leaving a $1/2 + 1/2$ in excess. Hence the triangle PSQ is also covered by the estimate $n/2$. This concludes the induction step and proves the lemma. $\square$

The remaining part of the solution in fact repeats the argument from the above proof. Consider the longest dissecting diagonal XY. Let $\mathcal{L}_{XY}$ be the shorter of the two parts of the boundary with endpoints X, Y and let XYZ be the triangle in the dissection with vertex Z not on $\mathcal{L}_{XY}$. Notice that XYZ is acute or right-angled, otherwise one of the segments XZ, YZ would be longer than XY. Denoting by $\mathcal{L}_{XZ}$, $\mathcal{L}_{YZ}$ the two pieces defined by Z and applying the lemma to each of $\mathcal{L}_{XY}$, $\mathcal{L}_{XZ}$, $\mathcal{L}_{YZ}$ we infer that there are no more than 2006/2 iso-good triangles in all, unless XYZ is one of them. But in that case XZ and YZ are good diagonals and the corresponding inequalities are strict. This shows that also in this case the total number of iso-good triangles in the dissection, including XYZ, is not greater than 1003.

This bound can be achieved. For this to happen, it just suffices to select a vertex of the 2006-gon and draw a broken line joining every second vertex, starting from the selected one. Since 2006 is even, the line closes. This already gives us the required 1003 iso-good triangles. Then we can complete the triangulation in an arbitrary fashion.

Problem 3.

Determine the least real number M such that the inequality

$$\left| ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2) \right| \leq M\left(a^2 + b^2 + c^2\right)^2$$

holds for all real numbers a, b and c.

Solution. We first consider the cubic polynomial

$$P(t) = tb(t^2 - b^2) + bc(b^2 - c^2) + ct(c^2 - t^2).$$

It is easy to check that $P(b) = P(c) = P(-b - c) = 0$, and therefore

$$P(t) = (b - c)(t - b)(t - c)(t + b + c),$$

since the cubic coefficient is $b-c$. The left-hand side of the proposed inequality can therefore be written in the form

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| = |P(a)| = |(b-c)(a-b)(a-c)(a+b+c)|.$$

The problem comes down to finding the smallest number M that satisfies the inequality

$$|(b-c)(a-b)(a-c)(a+b+c)| \le M\cdot(a^2+b^2+c^2)^2. \tag{1}$$

Note that this expression is symmetric, and we can therefore assume $a \le b \le c$ without loss of generality. With this assumption,

$$|(a-b)(b-c)| = (b-a)(c-b) \le \left(\frac{(b-a)+(c-b)}{2}\right)^2 = \frac{(c-a)^2}{4}, \tag{2}$$

with equality if and only if $b-a=c-b$, i.e. $2b=a+c$. Also

$$\left(\frac{(c-b)+(b-a)}{2}\right)^2 \le \frac{(c-b)^2+(b-a)^2}{2},$$

or equivalently,

$$3(c-a)^2 \le 2\cdot[(b-a)^2+(c-b)^2+(c-a)^2], \tag{3}$$

again with equality only for $2b=a+c$. From (2) and (3) we get

$$\begin{aligned}
&|(b-c)(a-b)(a-c)(a+b+c)|\\
\le\ & \frac{1}{4}\cdot|(c-a)^3(a+b+c)|\\
=\ & \frac{1}{4}\cdot\sqrt{(c-a)^6(a+b+c)^2}\\
\le\ & \frac{1}{4}\cdot\sqrt{\left(\frac{2\cdot[(b-a)^2+(c-b)^2+(c-a)^2]}{3}\right)^3\cdot(a+b+c)^2}\\
=\ & \frac{\sqrt{2}}{2}\cdot\left(\sqrt[4]{\left(\frac{(b-a)^2+(c-b)^2+(c-a)^2}{3}\right)^3\cdot(a+b+c)^2}\right)^2.
\end{aligned}$$

By the weighted AM-GM inequality this estimate continues as follows:

$$\begin{aligned}
&|(b-c)(a-b)(a-c)(a+b+c)|\\
\le\ & \frac{\sqrt{2}}{2}\cdot\left(\frac{(b-a)^2+(c-b)^2+(c-a)^2+(a+b+c)^2}{4}\right)^2\\
=\ & \frac{9\sqrt{2}}{32}\cdot(a^2+b^2+c^2)^2.
\end{aligned}$$

We see that the inequality (1) is satisfied for $M=\frac{9}{32}\sqrt{2}$, with equality if and only if $2b=a+c$ and

$$\frac{(b-a)^2+(c-b)^2+(c-a)^2}{3} = (a+b+c)^2.$$

Plugging $b=(a+c)/2$ into the last equation, we bring it to the equivalent form

$$2(c-a)^2 = 9(a+c)^2.$$

The conditions for equality can now be restated as

$$2b=a+c \qquad \text{and} \qquad (c-a)^2 = 18b^2.$$

Setting $b=1$ yields $a=1-\frac{3}{2}\sqrt{2}$ and $c=1+\frac{3}{2}\sqrt{2}$. We see that $M=\frac{9}{32}\sqrt{2}$ is indeed the smallest constant satisfying the inequality, with equality for any triple (a,b,c) proportional to $\left(1-\frac{3}{2}\sqrt{2},1,1+\frac{3}{2}\sqrt{2}\right)$, up to permutation.

Comment. With the notation $x = b-a$, $y = c-b$, $z = a-c$, $s = a+b+c$ and $r^2 = a^2+b^2+c^2$, the inequality (1) becomes just $|sxyz| \le Mr^4$ (with suitable constraints on s and r). The original asymmetric inequality turns into a standard symmetric one; from this point on the solution can be completed in many ways. One can e.g. use the fact that, for fixed values of $\sum x$ and $\sum x^2$, the product xyz is a maximum/minimum only if some of x, y, z are equal, thus reducing one degree of freedom, etc. A specific attraction of the problem is that the maximum is attained at a point (a, b, c) with all coordinates distinct.

Problem 4.

Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Solution. If (x, y) is a solution then obviously $x \ge 0$ and $(x, -y)$ is a solution too. For $x = 0$ we get the two solutions $(0, 2)$ and $(0, -2)$.

Now let (x, y) be a solution with $x > 0$; without loss of generality confine attention to $y > 0$. The equation rewritten as

$$2^x(1 + 2^{x+1}) = (y-1)(y+1)$$

shows that the factors $y-1$ and $y+1$ are even, exactly one of them divisible by 4. Hence $x \ge 3$ and one of these factors is divisible by 2^{x-1} but not by 2^x. So

$$y = 2^{x-1}m + \epsilon, \qquad m \text{ odd}, \qquad \epsilon = \pm 1. \tag{1}$$

Plugging this into the original equation we obtain

$$2^x\left(1 + 2^{x+1}\right) = \left(2^{x-1}m + \epsilon\right)^2 - 1 = 2^{2x-2}m^2 + 2^x m\epsilon,$$

or, equivalently

$$1 + 2^{x+1} = 2^{x-2}m^2 + m\epsilon.$$

Therefore

$$1 - \epsilon m = 2^{x-2}(m^2 - 8). \tag{2}$$

For $\epsilon = 1$ this yields $m^2 - 8 \le 0$, i.e., $m = 1$, which fails to satisfy (2).

For $\epsilon = -1$ equation (2) gives us

$$1 + m = 2^{x-2}(m^2 - 8) \ge 2(m^2 - 8),$$

implying $2m^2 - m - 17 \le 0$. Hence $m \le 3$; on the other hand m cannot be 1 by (2). Because m is odd, we obtain $m = 3$, leading to $x = 4$. From (1) we get $y = 23$. These values indeed satisfy the given equation. Recall that then $y = -23$ is also good. Thus we have the complete list of solutions (x, y): $(0, 2)$, $(0, -2)$, $(4, 23)$, $(4, -23)$.

Problem 5.

Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Solution. The claim is obvious if every integer fixed point of Q is a fixed point of P itself. For the sequel assume that this is not the case. Take any integer x_0 such that $Q(x_0) = x_0$, $P(x_0) \neq x_0$ and define inductively $x_{i+1} = P(x_i)$ for $i = 0, 1, 2, \ldots$; then $x_k = x_0$.

It is evident that

$$P(u) - P(v) \text{ is divisible by } u - v \text{ for distinct integers } u, v. \tag{1}$$

(Indeed, if $P(x) = \sum a_i x^i$ then each $a_i(u^i - v^i)$ is divisible by $u - v$.) Therefore each term in the chain of (nonzero) differences

$$x_0 - x_1, \quad x_1 - x_2, \quad \ldots, \quad x_{k-1} - x_k, \quad x_k - x_{k+1} \tag{2}$$

is a divisor of the next one; and since $x_k - x_{k+1} = x_0 - x_1$, all these differences have equal absolute values. For $x_m = \min(x_1, \ldots, x_k)$ this means that $x_{m-1} - x_m = -(x_m - x_{m+1})$. Thus $x_{m-1} = x_{m+1} (\neq x_m)$. It follows that consecutive differences in the sequence (2) have opposite signs. Consequently, $x_0, x_1, x_2, \ldots$ is an alternating sequence of two distinct values. In other words, every integer fixed point of Q is a fixed point of the polynomial $P(P(x))$. Our task is to prove that there are at most n such points.

Let a be one of them so that $b = P(a) \neq a$ (we have assumed that such an a exists); then $a = P(b)$. Take any other integer fixed point α of $P(P(x))$ and let $P(\alpha) = \beta$, so that $P(\beta) = \alpha$; the numbers α and β need not be distinct (α can be a fixed point of P), but each of α, β is different from each of a, b. Applying property (1) to the four pairs of integers (α, a), (β, b), (α, b), (β, a) we get that the numbers $\alpha - a$ and $\beta - b$ divide each other, and also $\alpha - b$ and $\beta - a$ divide each other. Consequently

$$\alpha - b = \pm(\beta - a), \qquad \alpha - a = \pm(\beta - b). \tag{3}$$

Suppose we have a plus in both instances: $\alpha - b = \beta - a$ and $\alpha - a = \beta - b$. Subtraction yields $a - b = b - a$, a contradiction, as $a \neq b$. Therefore at least one equality in (3) holds with a minus sign. For each of them this means that $\alpha + \beta = a + b$; equivalently $a + b - \alpha - P(\alpha) = 0$.

Denote $a + b$ by C. We have shown that every integer fixed point of Q other that a and b is a root of the polynomial $F(x) = C - x - P(x)$. This is of course true for a and b as well. And since P has degree $n > 1$, the polynomial F has the same degree, so it cannot have more than n roots. Hence the result.

Problem 6.

Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

Solution 1.

Lemma. Every convex $(2n)$-gon, of area S, has a side and a vertex that jointly span a triangle of area not less than S/n.

Proof. By *main diagonals* of the $(2n)$-gon we shall mean those which partition the $(2n)$-gon into two polygons with equally many sides. For any side b of the $(2n)$-gon denote by Δ_b the triangle ABP where A, B are the endpoints of b and P is the intersection point of the main diagonals AA', BB'. We claim that the union of triangles Δ_b, taken over all sides, covers the whole polygon.

To show this, choose any side AB and consider the main diagonal AA' as a directed segment. Let X be any point in the polygon, not on any main diagonal. For definiteness, let X lie on the left side of the ray AA'. Consider the sequence of main diagonals AA', BB', CC', $\ldots$, where $A, B, C, \ldots$ are consecutive vertices, situated right to AA'.

The n-th item in this sequence is the diagonal $A'A$ (i.e. AA' reversed), having X on its right side. So there are two successive vertices K, L in the sequence $A, B, C, \ldots$ before A' such that X still lies to the left of KK' but to the right of LL'. And this means that X is in the triangle $\Delta_{\ell'}$, $\ell' = K'L'$. Analogous reasoning applies to points X on the right of AA' (points lying on main diagonals can be safely ignored). Thus indeed the triangles Δ_b jointly cover the whole polygon.

The sum of their areas is no less than S. So we can find two opposite sides, say $b = AB$ and $b' = A'B'$ (with AA', BB' main diagonals) such that $[\Delta_b] + [\Delta_{b'}] \geq S/n$, where $[\cdots]$ stands for the area of a region. Let AA', BB' intersect at P; assume without loss of generality that $PB \geq PB'$. Then

$$[ABA'] = [ABP] + [PBA'] \geq [ABP] + [PA'B'] = [\Delta_b] + [\Delta_{b'}] \geq S/n,$$

proving the lemma. □

Now, let $\mathcal{P}$ be any convex polygon, of area S, with m sides $a_1, \ldots, a_m$. Let S_i be the area of the greatest triangle in $\mathcal{P}$ with side a_i. Suppose, contrary to the assertion, that

$$\sum_{i=1}^{m} \frac{S_i}{S} < 2.$$

Then there exist rational numbers $q_1, \ldots, q_m$ such that $\sum q_i = 2$ and $q_i > S_i/S$ for each i.

Let n be a common denominator of the m fractions $q_1, \ldots, q_m$. Write $q_i = k_i/n$; so $\sum k_i = 2n$. Partition each side a_i of $\mathcal{P}$ into k_i equal segments, creating a convex $(2n)$-gon of area S (with some angles of size 180°), to which we apply the lemma. Accordingly, this refined polygon has a side b and a vertex H spanning a triangle T of area $[T] \geq S/n$. If b is a piece of a side a_i of $\mathcal{P}$, then the triangle W with base a_i and summit H has area

$$[W] = k_i \cdot [T] \geq k_i \cdot S/n = q_i \cdot S > S_i,$$

in contradiction with the definition of S_i. This ends the proof.

Solution 2. As in the first solution, we allow again angles of size 180° at some vertices of the convex polygons considered.

To each convex n-gon $\mathcal{P} = A_1A_2 \ldots A_n$ we assign a centrally symmetric convex $(2n)$-gon $\mathcal{Q}$ with side vectors $\pm\overrightarrow{A_iA_{i+1}}$, $1 \leq i \leq n$. The construction is as follows. Attach the $2n$ vectors $\pm\overrightarrow{A_iA_{i+1}}$ at a common origin and label them $\overrightarrow{\mathbf{b}_1}, \overrightarrow{\mathbf{b}_2}, \ldots, \overrightarrow{\mathbf{b}_{2n}}$ in counterclockwise direction; the choice of the first vector $\overrightarrow{\mathbf{b}_1}$ is irrelevant. The order of labelling is well-defined if $\mathcal{P}$ has neither parallel sides nor angles equal to 180°. Otherwise several collinear vectors with the same direction are labelled consecutively $\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}, \ldots, \overrightarrow{\mathbf{b}_{j+r}}$. One can assume that in such cases the respective opposite vectors occur in the order $-\overrightarrow{\mathbf{b}_j}, -\overrightarrow{\mathbf{b}_{j+1}}, \ldots, -\overrightarrow{\mathbf{b}_{j+r}}$, ensuring that $\overrightarrow{\mathbf{b}_{j+n}} = -\overrightarrow{\mathbf{b}_j}$ for $j = 1, \ldots, 2n$. Indices are taken cyclically here and in similar situations below.

Choose points $B_1, B_2, \ldots, B_{2n}$ satisfying $\overrightarrow{B_jB_{j+1}} = \overrightarrow{\mathbf{b}_j}$ for $j = 1, \ldots, 2n$. The polygonal line $\mathcal{Q} = B_1B_2 \ldots B_{2n}$ is closed, since $\sum_{j=1}^{2n} \overrightarrow{\mathbf{b}_j} = \overrightarrow{0}$. Moreover, $\mathcal{Q}$ is a convex $(2n)$-gon due to the arrangement of the vectors $\overrightarrow{\mathbf{b}_j}$, possibly with 180°-angles. The side vectors of $\mathcal{Q}$ are $\pm\overrightarrow{A_iA_{i+1}}$, $1 \leq i \leq n$. So in particular $\mathcal{Q}$ is centrally symmetric, because it contains as side vectors $\overrightarrow{A_iA_{i+1}}$ and $-\overrightarrow{A_iA_{i+1}}$ for each $i = 1, \ldots, n$. Note that B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ are opposite sides of $\mathcal{Q}$, $1 \leq j \leq n$. We call $\mathcal{Q}$ the *associate* of $\mathcal{P}$.

Let S_i be the maximum area of a triangle with side A_iA_{i+1} in $\mathcal{P}$, $1 \leq i \leq n$. We prove that

$$[B_1B_2 \ldots B_{2n}] = 2\sum_{i=1}^{n} S_i \qquad (1)$$

and

$$[B_1B_2 \ldots B_{2n}] \geq 4\,[A_1A_2 \ldots A_n]\,. \qquad (2)$$

It is clear that (1) and (2) imply the conclusion of the original problem.

Lemma. For a side A_iA_{i+1} of $\mathcal{P}$, let h_i be the maximum distance from a point of $\mathcal{P}$ to line A_iA_{i+1}, $i = 1, \dots, n$. Denote by B_jB_{j+1} the side of $\mathcal{Q}$ such that $\overrightarrow{A_iA_{i+1}} = \overrightarrow{B_jB_{j+1}}$. Then the distance between B_jB_{j+1} and its opposite side in $\mathcal{Q}$ is equal to $2h_i$.

Proof. Choose a vertex A_k of $\mathcal{P}$ at distance h_i from line A_iA_{i+1}. Let $\mathbf{u}$ be the unit vector perpendicular to A_iA_{i+1} and pointing inside $\mathcal{P}$. Denoting by $\mathbf{x} \cdot \mathbf{y}$ the dot product of vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$h = \mathbf{u} \cdot \overrightarrow{A_iA_k} = \mathbf{u} \cdot (\overrightarrow{A_iA_{i+1}} + \dots + \overrightarrow{A_{k-1}A_k}) = \mathbf{u} \cdot (\overrightarrow{A_iA_{i-1}} + \dots + \overrightarrow{A_{k+1}A_k}).$$

In $\mathcal{Q}$, the distance H_i between the opposite sides B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ is given by

$$H_i = \mathbf{u} \cdot (\overrightarrow{B_jB_{j+1}} + \dots + \overrightarrow{B_{j+n-1}B_{j+n}}) = \mathbf{u} \cdot (\overrightarrow{\mathbf{b}_j} + \overrightarrow{\mathbf{b}_{j+1}} + \dots + \overrightarrow{\mathbf{b}_{j+n-1}}).$$

The choice of vertex A_k implies that the n consecutive vectors $\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}, \dots, \overrightarrow{\mathbf{b}_{j+n-1}}$ are precisely $\overrightarrow{A_iA_{i+1}}, \dots, \overrightarrow{A_{k-1}A_k}$ and $\overrightarrow{A_iA_{i-1}}, \dots, \overrightarrow{A_{k+1}A_k}$, taken in some order. This implies $H_i = 2h_i$. □

For a proof of (1), apply the lemma to each side of $\mathcal{P}$. If O the centre of $\mathcal{Q}$ then, using the notation of the lemma,

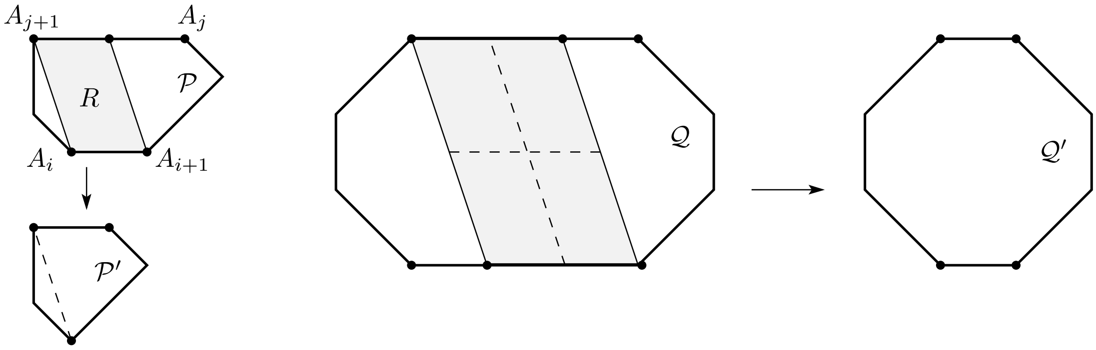
$$[B_jB_{j+1}O] = [B_{j+n}B_{j+n+1}O] = [A_iA_{i+1}A_k] = S_i\,.$$

Summation over all sides of $\mathcal{P}$ yields (1).

Set $d(\mathcal{P}) = [\mathcal{Q}] - 4[\mathcal{P}]$ for a convex polygon $\mathcal{P}$ with associate $\mathcal{Q}$. Inequality (2) means that $d(\mathcal{P}) \geq 0$ for each convex polygon $\mathcal{P}$. The last inequality will be proved by induction on the number ℓ of side directions of $\mathcal{P}$, i. e. the number of pairwise nonparallel lines each containing a side of $\mathcal{P}$.

We choose to start the induction with $\ell = 1$ as a base case, meaning that certain degenerate polygons are allowed. More exactly, we regard as *degenerate* convex polygons all closed polygonal lines of the form $X_1X_2\dots X_kY_1Y_2\dots Y_mX_1$, where $X_1, X_2, \dots, X_k$ are points in this order on a line segment X_1Y_1, and so are $Y_m, Y_{m-1}, \dots, Y_1$. The initial construction applies to degenerate polygons; their associates are also degenerate, and the value of d is zero. For the inductive step, consider a convex polygon $\mathcal{P}$ which determines ℓ side directions, assuming that $d(\mathcal{P}) \geq 0$ for polygons with smaller values of ℓ.

Suppose first that $\mathcal{P}$ *has a pair of parallel sides*, i. e. sides on distinct parallel lines. Let A_iA_{i+1} and A_jA_{j+1} be such a pair, and let $A_iA_{i+1} \leq A_jA_{j+1}$. Remove from $\mathcal{P}$ the parallelogram R determined by vectors $\overrightarrow{A_iA_{i+1}}$ and $\overrightarrow{A_iA_{j+1}}$. Two polygons are obtained in this way. Translating one of them by vector $\overrightarrow{A_iA_{i+1}}$ yields a new convex polygon $\mathcal{P}'$, of area $[\mathcal{P}] - [R]$ and with value of ℓ not exceeding the one of $\mathcal{P}$. The construction just described will be called operation **A**.



The associate of $\mathcal{P}'$ is obtained from $\mathcal{Q}$ upon decreasing the lengths of two opposite sides by an amount of $2A_iA_{i+1}$. By the lemma, the distance between these opposite sides is twice the distance between A_iA_{i+1} and A_jA_{j+1}. Thus operation **A** decreases $[\mathcal{Q}]$ by the area of a parallelogram with base and respective altitude twice the ones of R, i. e. by $4[R]$. Hence **A** leaves the difference $d(\mathcal{P}) = [\mathcal{Q}] - 4[\mathcal{P}]$ unchanged.

Now, if $\mathcal{P}'$ also has a pair of parallel sides, apply operation **A** to it. Keep doing so with the subsequent polygons obtained for as long as possible. Now, **A** decreases the number p of pairs of parallel sides in $\mathcal{P}$. Hence its repeated applications gradually reduce p to 0, and further applications of **A** will be impossible after several steps. For clarity, let us denote by $\mathcal{P}$ again the polygon obtained at that stage.

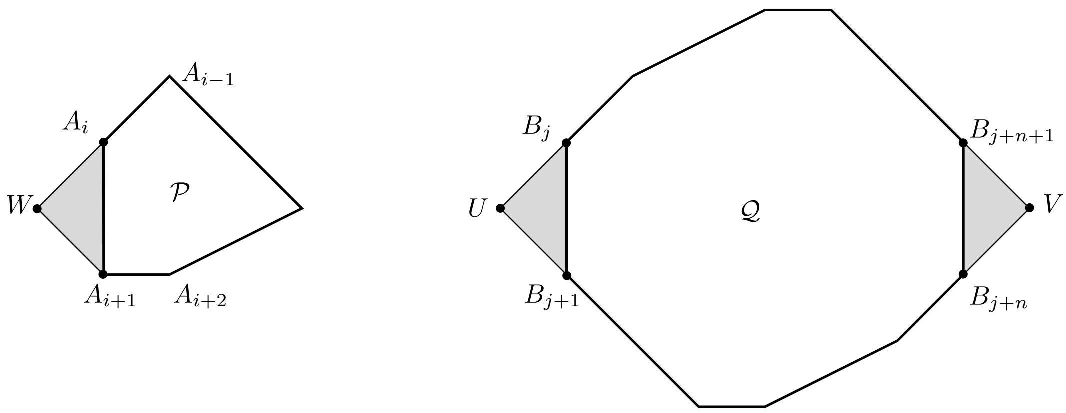
The inductive step is complete if $\mathcal{P}$ is degenerate. Otherwise $\ell > 1$ and $p = 0$, i. e. there are no parallel sides in $\mathcal{P}$. Observe that then $\ell \geq 3$. Indeed, $\ell = 2$ means that the vertices of $\mathcal{P}$ all lie on the boundary of a parallelogram, implying $p > 0$.

Furthermore, since $\mathcal{P}$ has no parallel sides, consecutive collinear vectors in the sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$ (if any) correspond to consecutive 180°-angles in $\mathcal{P}$. Removing the vertices of such angles, we obtain a convex polygon with the same value of $d(\mathcal{P})$.

In summary, if operation **A** is impossible for a nondegenerate polygon $\mathcal{P}$, then $\ell \geq 3$. In addition, one may assume that $\mathcal{P}$ has no angles of size 180°.

The last two conditions then also hold for the associate $\mathcal{Q}$ of $\mathcal{P}$, and we perform the following construction. Since $\ell \geq 3$, there is a side B_jB_{j+1} of $\mathcal{Q}$ such that the sum of the angles at B_j and B_{j+1} is greater than 180°. (Such a side exists in each convex k-gon for $k > 4$.) Naturally, $B_{j+n}B_{j+n+1}$ is a side with the same property. Extend the pairs of sides $B_{j-1}B_j, B_{j+1}B_{j+2}$ and $B_{j+n-1}B_{j+n}, B_{j+n+1}B_{j+n+2}$ to meet at U and V, respectively. Let $\mathcal{Q}'$ be the centrally symmetric convex $2(n{+}1)$-gon obtained from $\mathcal{Q}$ by inserting U and V into the sequence $B_1, \ldots, B_{2n}$ as new vertices between B_j, B_{j+1} and B_{j+n}, B_{j+n+1}, respectively. Informally, we adjoin to $\mathcal{Q}$ the congruent triangles $B_jB_{j+1}U$ and $B_{j+n}B_{j+n+1}V$. Note that B_j, B_{j+1}, B_{j+n} and B_{j+n+1} are kept as vertices of $\mathcal{Q}'$, although B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ are no longer its sides.

Let A_iA_{i+1} be the side of $\mathcal{P}$ such that $\overrightarrow{A_iA_{i+1}} = \overrightarrow{B_jB_{j+1}} = \overrightarrow{\mathbf{b}_j}$. Consider the point W such that triangle $A_iA_{i+1}W$ is congruent to triangle $B_jB_{j+1}U$ and exterior to $\mathcal{P}$. Insert W into the sequence $A_1, A_2, \ldots, A_n$ as a new vertex between A_i and A_{i+1} to obtain an $(n{+}1)$-gon $\mathcal{P}'$. We claim that $\mathcal{P}'$ is convex and its associate is $\mathcal{Q}'$.



Vectors $\overrightarrow{A_iW}$ and $\overrightarrow{\mathbf{b}_{j-1}}$ are collinear and have the same direction, as well as vectors $\overrightarrow{WA_{i+1}}$ and $\overrightarrow{\mathbf{b}_{j+1}}$. Since $\overrightarrow{\mathbf{b}_{j-1}}, \overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}$ are consecutive terms in the sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$, the angle inequalities $\angle(\overrightarrow{\mathbf{b}_{j-1}}, \overrightarrow{\mathbf{b}_j}) \leq \angle(\overrightarrow{A_{i-1}A_i}, \overrightarrow{\mathbf{b}_j})$ and $\angle(\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}) \leq \angle(\overrightarrow{\mathbf{b}_j}, \overrightarrow{A_{i+1}A_{i+2}})$ hold true. They show that $\mathcal{P}'$ is a convex polygon. To construct its associate, vectors $\pm\overrightarrow{A_iA_{i+1}} = \pm\overrightarrow{\mathbf{b}_j}$ must be deleted from the defining sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$ of $\mathcal{Q}$, and the vectors $\pm\overrightarrow{A_iW}$, $\pm\overrightarrow{WA_{i+1}}$ must be inserted appropriately into it. The latter can be done as follows:

$$\ldots,\ \overrightarrow{\mathbf{b}_{j-1}},\ \overrightarrow{A_iW},\ \overrightarrow{WA_{i+1}},\ \overrightarrow{\mathbf{b}_{j+1}},\ \ldots,\ -\overrightarrow{\mathbf{b}_{j-1}},\ -\overrightarrow{A_iW},\ -\overrightarrow{WA_{i+1}},\ -\overrightarrow{\mathbf{b}_{j+1}},\ \ldots .$$

This updated sequence produces $\mathcal{Q}'$ as the associate of $\mathcal{P}'$.

It follows from the construction that $[\mathcal{P}'] = [\mathcal{P}] + [A_iA_{i+1}W]$ and $[\mathcal{Q}'] = [\mathcal{Q}] + 2[A_iA_{i+1}W]$. Therefore $d(\mathcal{P}') = d(\mathcal{P}) - 2[A_iA_{i+1}W] < d(\mathcal{P})$.

To finish the induction, it remains to notice that the value of ℓ for $\mathcal{P}'$ is less than the one for $\mathcal{P}$. This is because side A_iA_{i+1} was removed. The newly added sides A_iW and WA_{i+1} do not introduce new side directions. Each one of them is either parallel to a side of $\mathcal{P}$ or lies on the line determined by such a side. The proof is complete.