The 40th Nordic Mathematical Contest
Solutions

27 March 2026

Problem 1 Let $n \geq 3$ be an integer. There are n knights sitting around a round table. On the table there are n candles, such that between each pair of adjacent knights there is a single candle. Some candles (possibly all, but possibly also none) are lit. For $i \in \{0,1,2\}$, let m_i be the number of knights sitting next to exactly i lit candles.

Find the smallest possible value of $m = \max\{m_0, m_1, m_2\}$ in terms of n.

Answer

$$m = \begin{cases} \frac{n}{3}+1, & \text{if } n \equiv 3 \pmod 6 \\ \left\lceil \frac{n}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

Solution We prove that the above is the answer. As $m_0 + m_1 + m_2 = n$ the pigeonhole principle implies $m_i \geq \frac{n}{3}$ for some $i \in \{0,1,2\}$. Hence $m \geq \left\lceil \frac{n}{3} \right\rceil$. This provides the lower bound for $n \neq 6k+3$.

We now prove the lower bound for $n = 6k+3$.

We claim that m_1 is always even. Walk around the table and put a $+1$ if you go from an unlit candle to a lit candle and -1 if you go from a lit candle to an unlit candle. After a full round the sum of the $+1$'s and -1's must equal 0. Hence the number of $+1$'s equals the number of -1's. Therefore the total number of $+1$'s and -1's is even, but this equals m_1.

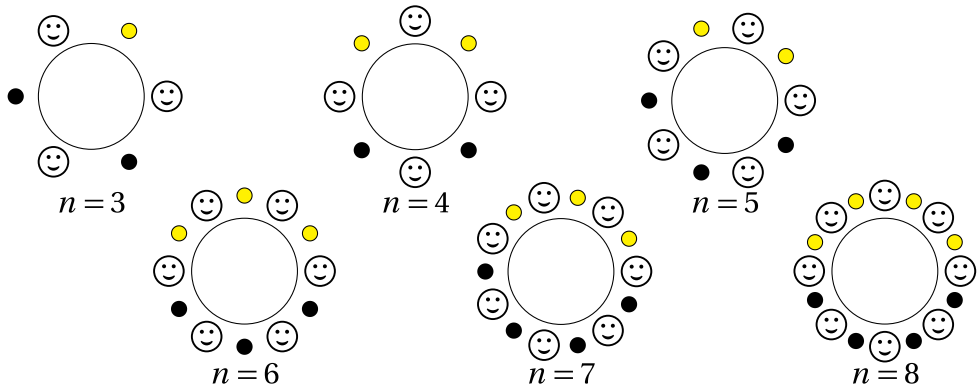
Assume that $n = 6k+3$ for $k \in \mathbb{N}$. Then

$$m_0 + m_1 + m_2 = 6k+3.$$

If $m_1 \leq 2k+1$ we must have $m_1 \leq 2k$ as m_1 is even. Thus $m_0 + m_2 \geq 4k+3$. By the pigeonhole principle $m \geq 2k+2$. If on the other hand $m_1 \geq 2k+1$ we must have $m_1 \geq 2k+2$. In both cases we have

$$m \geq 2k+2 = \frac{n}{3}+1.$$

We now need to prove that these lower bounds are attainable. We do this by induction on n, with base cases $n = 3,4,5,6,7,8$ drawn below:



For the induction step we go from n to $n+6$. We note that in all configurations we have a pair of unlit candles next to each other. Between this pair squeeze in another 6 people numbered $1,\dots,6$. Then light the candles between the pairs $(1,2)$, $(2,3)$, and $(3,4)$. Note that m_i is increased by 2 in this step for all $i \in \{0,1,2\}$ and hence m is also increased by two. As there is still a pair of unlit candles next to each other this finishes the induction.

Problem 2 Consider the system of equations:

$$\begin{cases} x^2 = y+1, \\ xy = x+y. \end{cases}$$

Show that if $(x,y)=(x_0,y_0)$, where x_0 and y_0 are real numbers, is a solution to the above system of equations, then there is also a solution $(x,y)=(x_1,y_1)$, where x_1 and y_1 are real numbers and $y_1 x_0 = 1$.

Solution Assume that $(x,y)=(x_0,y_0)$ is a solution. Note that none of x_0, y_0 can be 0 or 1. From the second equation we get

$$y_0 = \frac{x_0}{x_0-1},$$

which plugged into the first equation gives

$$x_0^2 = \frac{x_0}{x_0-1}+1.$$

This rewrites to

$$1 = -x_0^3 + x_0^2 + 2x_0. \tag{1}$$

Let $y_1 = \frac{1}{x_0}$ and let

$$x_1 = \frac{y_1}{y_1 - 1} = \frac{1}{1 - x_0}.$$

We claim that the pair (x_1, y_1) satisfies the system of equations. It follows from the definition of x_1 that it satisfies the second equation. Now note

$$x_1^2 - 1 = \frac{1}{(1-x_0)^2} - \frac{(1-x_0)^2}{(1-x_0)^2} = \frac{-x_0^2 + 2x_0}{x_0^2 + 1 - 2x_0}.$$

Using equation (1) we obtain

$$x_1^2 - 1 = \frac{-x_0^2 + 2x_0}{x_0^2 + 1 - 2x_0} = \frac{-x_0^2 + 2x_0}{x_0^2 + \left(-x_0^3 + x_0^2 + 2x_0\right) - 2x_0} = \frac{-x_0^2 + 2x_0}{-x_0^3 + 2x_0^2} = \frac{1}{x_0} = y_1.$$

Thus (x_1, y_1) satisfies the system of equations.

Problem 3 Let $ABCD$ be a convex quadrilateral such that $BA = BC$. The internal angle bisectors of $\angle DBA$ and $\angle CBD$ intersect the perpendicular bisectors of AD and CD in E and F, respectively.

Prove that the circumcircles of $\triangle DAC$ and $\triangle DEF$ are tangent.

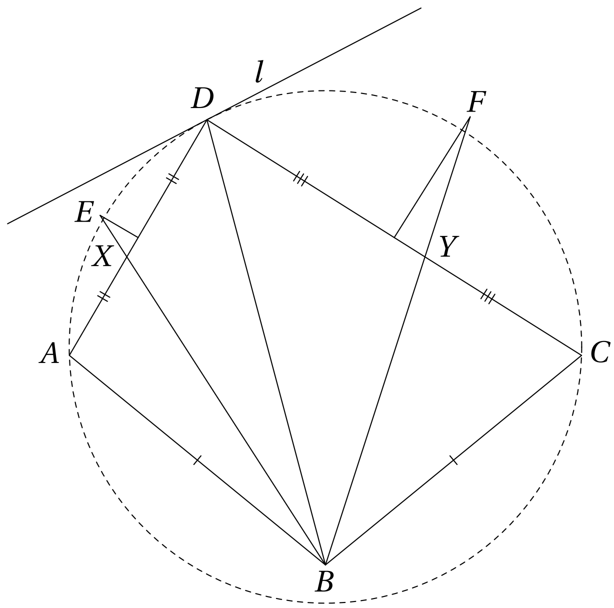
Note: A convex quadrilatral is a quadrilateral where all angles are smaller than 180°.

Note: The internal angle bisector of an angle is the line segment that divides the angle into two equal parts.

Solution Let $X = BE \cap AD$ and $Y = CD \cap BF$. From the angle bisector theorem we have

$$\frac{AX}{XD} = \frac{AB}{BD} = \frac{BC}{BD} = \frac{CY}{YD},$$

so $XY \parallel AC$.



By definition of E, we see that $DEAB$ is cyclic with E as the midpoint of arc AD. Similarly, $DFCB$ is cyclic with F being the midpoint of arc CD. Hence $\triangle ABX \sim \triangle EBD$ and $\triangle CBY \sim \triangle FBD$, whence

$$BX \cdot BE = AB \cdot BD = BC \cdot BD = BY \cdot BF,$$

so $XYFE$ is cyclic. Let l be the tangent to (DAC) at D. We compute

$$\begin{aligned}
\angle(DE, l) = \angle(DA, l) - \angle EDA &= \angle DCA - \angle EBA \\
&= \angle DCA + \angle BAC - \angle(AC, BE) \\
&= \angle DCA + \angle ACB - \angle EFB \\
&= \angle DCB - \angle EFB \\
&= \angle DFB - \angle EFB = \angle DFE,
\end{aligned}$$

so l is also tangent to (DEF).

Problem 4 A pair of positive integers (a,b) is *good* if all the fractions

$$\frac{a}{b}, \frac{a+1}{b+1}, \cdots, \frac{a+9}{b+9}$$

are integers.

a) Prove that there are only finitely many good pairs (a,b) with $b < a < b^9$.

b) Prove that there are infinitely many good pairs (a,b) with $b < a < b^{10}$.

Solution For part a) suppose that (a,b) is a good pair. This implies $a-b$ must be divisible by $b, b+1, \ldots, b+9$. If we let M be the lowest common multiple of $b, b+1, \ldots, b+9$, then M divides $a-b$. We may write

$$M = \prod_p p^{m_p},$$

as a product of primes. For a prime p let n_p be the greatest number such that

$$p^{n_p} | b(b+1)(b+2)\cdots(b+9).$$

We want to compare n_p to m_p for each prime p.

- If $p \geq 11$ then p divides at most one of $b, b+1, \ldots, b+9$ and hence $n_p = m_p$.
- If $p = 2$ then exactly 5 of the numbers $b, b+1, \ldots, b+9$ are divisible by p and at most one of them is divisible by 2^4. Thus

$$n_2 \leq m_2 + (5-1)(4-1) = m_2 + 12.$$

- If $p = 3$ then at most 4 of the numbers $b, b+1, \ldots, b+9$ are divisible by p and at most one of them is divisible by 3^3. Thus

$$n_3 \leq m_3 + (4-1)(3-1) = m_3 + 6.$$

- If $p = 5$ or 7 at most 2 of the numbers $b, b+1, \ldots, b+9$ are divisible by p and at most one of them is divisible by p^2. Thus

$$n_p \leq m_p + 1.$$

Now it follows that

$$a-b \geq M = \prod_p p^{m_p} \geq \frac{\prod_p p^{n_p}}{2^{12}\cdot 3^6 \cdot 5^1 \cdot 7^1} = \frac{b(b+1)\cdots(b+9)}{104509440} > \frac{b^{10}}{104509440}.$$

If $b > 104509440$, then $a > b^9$. Therefore, there are only finitely many good pairs (a,b) such that $b < a < b^9$ as there is only finitely many choices for b.

For part b) we may take

$$a = b + \frac{b(b+1)\cdots(b+9)}{2},$$

which is less than b^{10} if b is sufficiently large. Thus we get infinitely many good pairs (a,b) with $b < a < b^{10}$.