English version

Time allowed is 4 hours. Each problem is worth 7 points.
The only permitted aids are writing and drawing materials.

Problem 1

Let $T(a)$ be the sum of digits of a. For which positive integers R does there exist a positive integer n such that $\frac{T(n^2)}{T(n)} = R$?

Problem 2

Let $\mathcal{Q}_1$ be a quadrilateral such that the midpoints of its sides lie on a circle. Prove that there exists a cyclic quadrilateral $\mathcal{Q}_2$ with the same sidelengths as $\mathcal{Q}_1$, such that two of the angles in $\mathcal{Q}_2$ are equal.

Problem 3

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)f(y) + y) = f(x)y + f(y - x + 1)$$

for all $x, y \in \mathbb{R}$.

Problem 4

Alice and Bob are playing a game. First, Alice chooses a partition $\mathcal{C}$ of the positive integers, i.e. a (not necessarily finite) set of subsets of the positive integers such that each positive integer is in exactly one of the sets in $\mathcal{C}$. Then Bob does the following operation a finite number of times.

Choose a set $S \in \mathcal{C}$ not previously chosen, and let D be the set of all positive integers dividing at least one element in S. Then add the set $D \setminus S$ (possibly the empty set) to $\mathcal{C}$.

Bob wins if in any of the operations the set $D \setminus S$ is already in $\mathcal{C}$, otherwise, Alice wins. Determine which player has a winning strategy.

Note: $D \setminus S$ is the set of all elements in D that are not in S.