

# The 37th Nordic Mathematical Contest

Thursday, 30 March 2023

English version

*Time allowed: 4 hours. Each problem is worth 7 points.  
Only writing and drawing tools are allowed.*

**Problem 1** Alice and Bianca have one hundred marbles. At the start of the game they split these hundred marbles into two piles. Thereafter, a move consists of choosing a pile, then choosing a positive integer not larger than half of the number of marbles in that pile, and finally removing that number of marbles from the chosen pile. The first player unable to remove any marbles loses. Alice makes the first move of the game. Determine all initial pile sizes for which Bianca has a winning strategy.

**Problem 2** Let  $\mathbb{N}_+$  denote the set of positive integers. Find all functions  $f : \mathbb{N}_+ \rightarrow \mathbb{N}_+$  such that

$$\gcd(f(x), y)f(xy) = f(x)f(y)$$

for all  $x, y \in \mathbb{N}_+$ .

**Problem 3** Find all sequences of integers  $a_0, a_1, a_2, \dots$  such that for any integers  $k, \ell \geq 0$ , we have

$$a_k - a_\ell | k^2 - \ell^2,$$

that is, for any integers  $k, \ell \geq 0$ , there exists some integer  $z$  such that  $(a_k - a_\ell)z = k^2 - \ell^2$ .

**Problem 4** Let  $ABC$  be a triangle, and  $M$  the midpoint of the side  $BC$ . Let  $E$  and  $F$  be points on the sides  $AC$  and  $AB$ , respectively, so that  $ME = MF$ . Let  $D$  be the second intersection of the circumcircle of  $MEF$  and the side  $BC$ . Consider the lines  $\ell_D, \ell_E$  and  $\ell_F$  through  $D, E$  and  $F$ , respectively, such that  $\ell_D \perp BC, \ell_E \perp CA$  and  $\ell_F \perp AB$ . Show that  $\ell_D, \ell_E$  and  $\ell_F$  are concurrent.