The 28th Nordic Mathematical Contest

Monday, 31 March 2014

Problem set with solutions

The time allowed is 4 hours. Each problem is worth 5 points.
The only permitted aids are writing and drawing tools.

Problem 1

Find all functions $f : \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ is the set of the natural numbers and is assumed to contain 0), such that

$$f(x^2) - f(y^2) = f(x+y)f(x-y),$$

for all $x, y \in \mathbb{N}$ with $x \geq y$.

Problem 2

Given an equilateral triangle, find all points inside the triangle such that the distance from the point to one of the sides is equal to the geometric mean of the distances from the point to the other two sides of the triangle.

[The geometric mean of two numbers x and y equals $\sqrt{xy}$.]

Problem 3

Find all nonnegative integers a, b, c, such that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} = \sqrt{2014}.$$

Problem 4

A game is played on an $n \times n$ chessboard. At the beginning there are 99 stones on each square. Two players A and B take turns, where in each turn the player chooses either a row or a column and removes one stone from each square in the chosen row or column. They are only allowed to choose a row or a column, if it has least one stone on each square. The first player who cannot move, looses the game. Player A takes the first turn. Determine all n for which player A has a winning strategy.

Solution 1

It is easily seen that both $f(x) = x$ and $f \equiv 0$ solve the equation; we shall show that there are no other solutions.

Setting $x = y = 0$ gives $f(0) = 0$; if only $y = 0$ we get $f(x^2) = (f(x))^2$, for all admissible x. For $x = 1$ we now get $f(1) = 0$, or $f(1) = 1$.

Case 1. $f(1) = 0$: We have

$$f((x+1)^2) - f(x^2) = f(2x+1) \cdot f(1) = 0 = (f(x+1))^2 - (f(x))^2,$$

so that $f(x+1) = f(x)$ for all x, and it follows that $f \equiv 0$.

Case 2. $f(1) = 1$: Denote $f(2) = a$. We have

$$(f(2))^2 - 1 = f(2^2) - f(1^2) = f(3) \cdot f(1),$$

so that $f(3) = a^2 - 1$. Obviously $f(4) = a^2$, and $x = 3$, $y = 1$ now give

$$(a^2 - 1)^2 - 1 = a^3,$$

so that $a = 0$ or $a = 2$, since a cannot be negative. If $f(2) = 0$, then $f(3) = 0 - 1$, which is impossible. Thus we have $a = 2$. The fact that $f(n) = n$ for all $n \in \mathbb{N}$ is now easy to establish using induction.

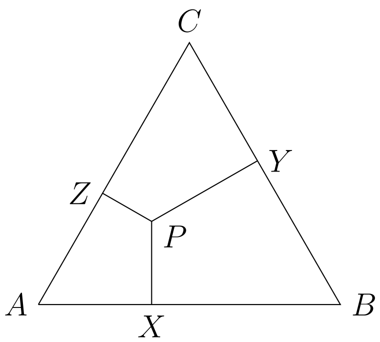
Solution 2

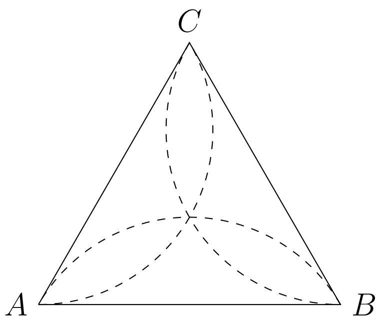
Let P be a point inside $\triangle ABC$. Denote its orthogonal projections on AB, BC, CA by X, Y, Z, respectively. We have $\angle XPZ = \angle YPX = 120°$.

Assume that $PX^2 = PY \cdot PZ$. Together with $\angle XPZ = \angle YPX = 120°$, this gives $\triangle XPZ \sim \triangle YPX$ (s-a-s). It means that $\angle PZX = \angle PXY$. The quadrilaterals $AXPZ$ and $BYPX$ are circumscribed, and we get $\angle PAX = \angle PBY$, so that $\angle PAB + \angle PBA = 60°$. We now have $\angle APB = 120°$, meaning that P lies on an arc inside the triangle, which is part of the circle through A, B, and the centre of the triangle.

The above argument can be reversed to see that all points on this arc satisfy the condition.

The set of all points as described is thus the union of three arcs, each of them passing through two of the vertices and through the centre of the triangle.





Remark: It is also possible to solve this by introducing a coordinate system and deriving equations for the locus of P.

Solution 3

We start with a lemma:

Lemma. If p, q are nonnegative integers and $\sqrt{p} + \sqrt{q} = r \in \mathbb{Q}$, then p and q are squares of integers.

Proof of lemma: If $r = 0$, then $p = q = 0$. For $r \neq 0$, take the square of both sides to get $p + q + 2\sqrt{pq} = r^2$, which means that $\sqrt{pq} \in \mathbb{Q}$, so that pq must be the square of a rational number, and, being an integer, it must be the square of an integer. Denote $pq = s^2$, $s \geq 0$. Then $p = \dfrac{s^2}{q}$, and

$$r = \sqrt{p} + \sqrt{q} = \frac{s}{\sqrt{q}} + \sqrt{q},$$

which implies that $\sqrt{q} = \dfrac{s+q}{r} \in \mathbb{Q}$, and it follows that q is a square. Then p must also be a square.

Back to the problem: we can rewrite the equation as

$$a + b + 2\sqrt{ab} = 2014 + c - 2\sqrt{2014c},$$

so that

$$\sqrt{ab} + \sqrt{2014c} \in \mathbb{Q}.$$

The lemma now tells us that ab and $2014c$ need to be squares of integers. Since $2014 = 2 \cdot 19 \cdot 53$, we must have $c = 2014m^2$ for some nonnegative integer m. Similarly, $a = 2014k^2$, $b = 2014l^2$. The equation now implies

$$k + l + m = 1,$$

so that the only possibilities are $(2014, 0, 0)$, $(0, 2014, 0)$, $(0, 0, 2014)$.

Solution 4

Player A has a winning strategy if and only if n is odd.

First we prove that no matter how they play, the play will not end before the board is empty. Let (i, j) denote the square in row i and column j, let r_i denote the number of times row i has been chosen when the game ends, and let c_j denote the same for columns. Assume by contradiction that there is a none empty square (a, b) when no more moves are possible. Hence there is an empty square in row a, let us say (a, c), and an empty square in column b, let us say (d, b). This shows that $r_a + c_b < 99$, $r_a + c_c = 99$ and $r_d + c_b = 99$. But this leads to $r_d + c_c > 99$ which is impossible since there are exactly 99 stones on square (d, c) when the game begins.

This shows that the game will end after $\frac{n \times n \times 99}{99} = n \times n$ moves since each player removes 99 stones in each move. The number $n \times n$ has the same parity as n, and hence A wins if n is odd and B wins if n is even no matter how they play.

Remark: It can by shown that player B has a winning strategy when n is even in a very different way: If player B copies the choice of A, i.e. when A chooses row m, B chooses row $n - m$, and the same for columns, then player B wins when n is even.