The 27^{th} Nordic Mathematical Contest

Monday, 8 April 2013

English Version

The time allowed is 4 hours. Each problem is worth 5 points.
The only permitted aids are writing and drawing tools.

PROBLEM 1. Let $(a_n)_{n\geq 1}$ be a sequence with $a_1 = 1$ and

$$a_{n+1} = \left\lfloor a_n + \sqrt{a_n} + \tfrac{1}{2} \right\rfloor$$

for all $n \geq 1$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Find all $n \leq 2013$ such that a_n is a perfect square.

PROBLEM 2. In a football tournament there are n teams, with $n \geq 4$, and each pair of teams meets exactly once. Suppose that, at the end of the tournament, the final scores form an arithmetic sequence where each team scores 1 more point than the following team on the scoreboard. Determine the maximum possible score of the lowest scoring team, assuming usual scoring for football games (where the winner of a game gets 3 points, the loser 0 points, and if there is a tie both teams get 1 point).

PROBLEM 3. Define a sequence $(n_k)_{k\geq 0}$ by $n_0 = n_1 = 1$, and $n_{2k} = n_k + n_{k-1}$ and $n_{2k+1} = n_k$ for $k \geq 1$. Let further $q_k = n_k/n_{k-1}$ for each $k \geq 1$. Show that every positive rational number is present exactly once in the sequence $(q_k)_{k\geq 1}$.

PROBLEM 4. Let ABC be an acute angled triangle, and H a point in its interior. Let the reflections of H through the sides AB and AC be called H_c and H_b, respectively, and let the reflections of H through the midpoints of these same sides be called H'_c and H'_b, respectively. Show that the four points H_b, H'_b, H_c, and H'_c are concyclic if and only if at least two of them coincide or H lies on the altitude from A in triangle ABC.