The 27^{th} Nordic Mathematical Contest

Monday, 8 April 2013

Solution

Each problem is worth 5 points.

Problem 1. Let $(a_n)_{n\geq 1}$ be a sequence with $a_1 = 1$ and

$$a_{n+1} = \left\lfloor a_n + \sqrt{a_n} + \tfrac{1}{2} \right\rfloor$$

for all $n \geq 1$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Find all $n \leq 2013$ such that a_n is a perfect square.

Solution. We will show by induction that $a_n = 1 + \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor$, which is equivalent to $a_{2m} = 1 + m^2$ and $a_{2m+1} = 1 + m(m+1)$. Clearly this is true for a_1.
If $a_{2m+1} = 1 + m(m+1)$ then

$$a_{2m+2} = \left\lfloor m^2 + m + 1 + \sqrt{m^2 + m + 1} + \tfrac{1}{2} \right\rfloor,$$

and since $m + \frac{1}{2} < \sqrt{m^2 + m + 1} < m + 1$ (easily seen by squaring), we get $a_{2m+2} = (m^2 + m + 1) + (m + 1) = 1 + (m + 1)^2$.
And if $a_{2m} = 1 + m^2$ then

$$a_{2m+1} = \left\lfloor m^2 + 1 + \sqrt{m^2 + 1} + \tfrac{1}{2} \right\rfloor,$$

and here $m < \sqrt{m^2 + 1} < m + \frac{1}{2}$, so $a_{2m+1} = (m^2 + 1) + m = 1 + m(m + 1)$.
If $m \geq 1$ then $m^2 < 1 + m^2 < (m + 1)^2$ and $m^2 < m^2 + m + 1 < (m + 1)^2$, so a_n cannot be a perfect square if $n > 1$. Therefore $a_1 = 1$ is the only perfect square in the sequence.

Problem 2. In a football tournament there are n teams, with $n \geq 4$, and each pair of teams meets exactly once. Suppose that, at the end of the tournament, the final scores form an arithmetic sequence where each team scores 1 more point than the following team on the scoreboard. Determine the maximum possible score of the lowest scoring team, assuming usual scoring for football games (where the winner of a game gets 3 points, the loser 0 points, and if there is a tie both teams get 1 point).

Solution. Note that the total number of games equals the number of different pairings, that is, $n(n-1)/2$. Suppose the lowest scoring team ends with k points. Then the total score for all teams is

$$k + (k + 1) + \cdots + (k + n - 1) = nk + \frac{(n-1)n}{2}.$$

Some games must end in a tie, for otherwise, all team scores would be a multiple of 3 and cannot be 1 point apart. Since the total score of a tie is only 2 points compared to 3 points if one of the teams wins, we therefore know that

$$nk + \frac{(n-1)n}{2} < 3 \cdot \frac{n(n-1)}{2},$$

so $nk < n(n-1)$, and hence $k < n-1$. It follows that the lowest scoring team can score no more than $n-2$ points.

We now show by induction that it is indeed possible for the lowest scoring team to score $n-2$ points.

The following scoreboard shows this is possible for $n = 4$:

$$\begin{array}{cccc|c} - & 3 & 1 & 1 & 5 \\ 0 & - & 1 & 3 & 4 \\ 1 & 1 & - & 1 & 3 \\ 1 & 0 & 1 & - & 2 \end{array}$$

Now suppose we have a scoreboard for n teams labelled $T_{n-2}, \ldots, T_{2n-3}$, where team T_i scores i points. Keep the results among these teams unchanged while adding one more team.

Write $n = 3q + r$ with $r \in \{1, -1, 0\}$, and let the new team tie with just one of the original teams, lose against q teams, and win against the rest of them. The new team thus wins $n-1-q$ games, and gets $1 + 3(n-1-q) = 3n-2-3q = 2n-2+r$ points.

Moreover, we arrange for the q teams which win against the new team to form an arithmetic sequence T_j, T_{j+3}, $\ldots$, $T_{j+3(q-1)} = T_{j+n-r-3}$, so that each of them, itself having gained three points, fills the slot vacated by the next one.

(i) If $r = 1$, then let the new team tie with team T_{n-2} and lose to each of the teams $T_{n-1}, T_{n+2}, \ldots, T_{n-1+n-r-3} = T_{2n-5}$.

Team T_{n-2} now has $n-1$ points and takes the place vacated by T_{n-1}. At the other end, T_{2n-5} now has $2n-2$ points, just one more than the previous top team T_{2n-3}. And the new team has $2n-2+r = 2n-1$ points, becoming the new top team. The teams now have all scores from $n-1$ up to $2n-1$.

(ii) If $r = -1$, then let the new team tie with team T_{2n-3} and lose to each of the teams $T_{n-2}, T_{n+1}, \ldots, T_{n-2+n-r-3} = T_{2n-4}$.

The old top team T_{2n-3} now has $2n-2$ points, and its former place is filled by the new team, which gets $2n-2+r = 2n-3$ points. T_{2n-4} now has $2n-1$ points and is the new top team. So again we have all scores ranging from $n-1$ up to $2n-1$.

(iii) If $r = 0$, then let the new team tie with team T_{n-2} and lose to teams $T_{n-1}, T_{n+2}, \ldots, T_{n-1+n-r-3} = T_{2n-4}$.

Team T_{n-2} now has $n-1$ points and fills the slot vacated by T_{n-1}. At the top end, T_{2n-4} now has $2n-1$ points, while the new team has $2n-2+r = 2n-2$ points, and yet again we have all scores from $n-1$ to $2n-1$.

This concludes the proof.

See next page for problem 3.

PROBLEM 3. Define a sequence $(n_k)_{k\geq 0}$ by $n_0 = n_1 = 1$, and $n_{2k} = n_k + n_{k-1}$ and $n_{2k+1} = n_k$ for $k \geq 1$. Let further $q_k = n_k/n_{k-1}$ for each $k \geq 1$. Show that every positive rational number is present exactly once in the sequence $(q_k)_{k\geq 1}$.

SOLUTION. Clearly, all the numbers n_k are positive integers. Moreover,

$$q_{2k} = \frac{n_{2k}}{n_{2k-1}} = \frac{n_k + n_{k-1}}{n_{k-1}} = q_k + 1, \tag{1}$$

and similarly,

$$\frac{1}{q_{2k+1}} = \frac{n_{2k}}{n_{2k+1}} = \frac{n_k + n_{k-1}}{n_k} = \frac{1}{q_k} + 1. \tag{2}$$

In particular, $q_k > 1$ when k is even, and $q_k < 1$ when $k \geq 3$ is odd.

We will show the following by induction on $t = 2, 3, 4, \ldots$:

CLAIM: *Every rational number r/s where r, s are positive integers with $\gcd(r, s) = 1$ and $r + s \leq t$ occurs precisely once among the numbers q_k.*

The claim is clearly true for $t = 2$, since then $r/s = 1/1 = 1$ is the only possibility, and q_1 is the only occurrence of 1 in the sequence.

Now, assume that $u \geq 3$ and that the claim holds for $t = u - 1$. Let r and s be positive integers with $\gcd(r, s) = 1$ and $r + s = u$.

First, assume that $r > s$. We know that $r/s = q_m$ is only possible if m is even. But

$$\frac{r}{s} = q_{2k} \Leftrightarrow \frac{r-s}{s} = q_k$$

by (1), and moreover, the latter equality holds for precisely one k according to the induction hypothesis, since $\gcd(r - s, s) = 1$ and $(r - s) + s = r \leq t$.

Next, assume that $r < s$. We know that $r/s = q_m$ is only possible if m is odd. But

$$\frac{r}{s} = q_{2k+1} \Leftrightarrow \frac{s}{r} = \frac{1}{q_{2k+1}} \Leftrightarrow \frac{s-r}{r} = \frac{1}{q_k}$$

by (2), and moreover, the latter equality holds for precisely one k according to the induction hypothesis, since $\gcd(s - r, r) = 1$ and $(s - r) + r = s \leq t$.

See next page for problem 4.

Problem 4. Let ABC be an acute angled triangle, and H a point in its interior. Let the reflections of H through the sides AB and AC be called H_c and H_b, respectively, and let the reflections of H through the midpoints of these same sides be called H'_c and H'_b, respectively. Show that the four points H_b, H'_b, H_c, and H'_c are concyclic if and only if at least two of them coincide or H lies on the altitude from A in triangle ABC.

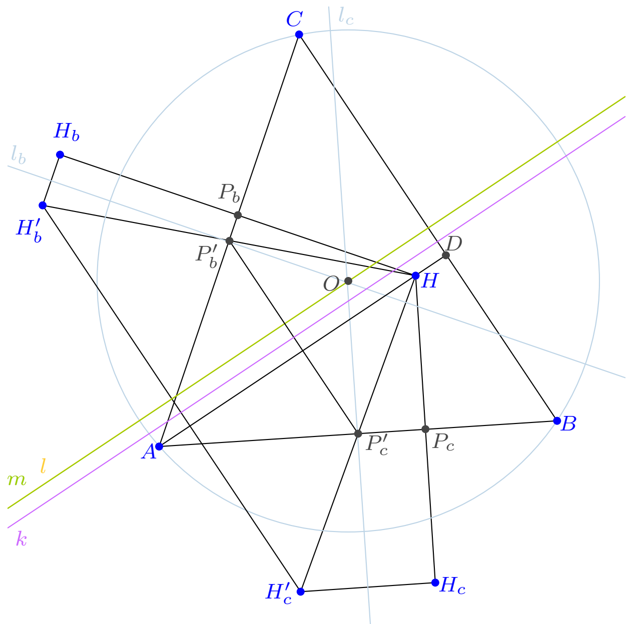
Solution. If at least two of the four points H_b, H'_b, H_c, and H'_c coincide, all four are obviously concyclic. Therefore we may assume that these four points are distinct.

Let P_b denote the midpoint of segment HH_b, P'_b the midpoint of segment HH'_b, P_c the midpoint of segment HH_c, and P'_c the midpoint of segment HH'_c.

The triangle $HH_bH'_b$ being right-angled in H_b, it follows that the perpendicular bisector ℓ_b of the side $H_bH'_b$ goes through the point P'_b. Since the segments $P_bP'_b$ and $H_bH'_b$ are parallel and P'_b is the midpoint of the side AC, we then conclude that ℓ_b also goes through the circumcentre O of triangle ABC.

Similarly the perpendicular bisector ℓ_c of the segment $H_cH'_c$ also goes through O. Hence the four points H_b, H'_b, H_c, and H'_c are concyclic if and only if also the perpendicular bisector ℓ of the segment $H'_bH'_c$ goes through the point O. Since $H'_bH'_c \parallel P'_bP'_c \parallel BC$, this is the case if and only if ℓ is the perpendicular bisector m of the segment BC.

Let k denote the perpendicular bisector of the segment $P'_bP'_c$. Since the lines ℓ and m are obtained from k by similarities of ratio 2 and centres H and A, respectively, they coincide if and only if HA is parallel to m. Thus H_b, H'_b, H_c, and H'_c are concyclic if and only if H lies on the altitude from A in triangle ABC.



Click here to experiment with the figure in GeoGebra.