15th Nordic Mathematical Contest

Thursday March 29th, 2001

English version

Time allowed: 4 hours. Each problem is worth 5 points.

Problem 1. Let A be a finite collection of squares in a coordinate plane such that each square in A has for its corners points of the form (m, n), $(m+1, n)$, $(m, n+1)$ and $(m+1, n+1)$ for some integers m and n.

Show that there exists a subcollection B of A consisting of at least 25% of all the squares in A such that no two distinct squares in B have a common corner point.

Problem 2. Let f be a bounded real-valued function defined for all real values such that the following condition is satisfied for every real number x:

$$f(x+\tfrac{1}{3}) + f(x+\tfrac{1}{2}) = f(x) + f(x+\tfrac{5}{6})$$

Show that f is periodic. (A function f is called periodic, if there exists a positive number k, such that $f(x+k) = f(x)$ for every real number x).

Problem 3. Determine the number of real roots in the equation

$$x^8 - x^7 + 2x^6 - 2x^5 + 3x^4 - 3x^3 + 4x^2 - 4x + \tfrac{5}{2} = 0$$

Problem 4. Let $ABCDEF$ be a convex hexagon in which each of the diagonals AD, BE and CF divides the hexagon in two quadrilaterals with equal areas.

Show that AD, BE and CF pass through the same point.