10th Nordic Mathematical Contest

Thursday, March 11, 1996

Version: English

1. Prove the existence of a positive integer divisible by 1996 the sum of whose decimal digits is 1996.

2. Determine all real x such that
$$x^n + x^{-n}$$
is an integer for any integer n.

3. A circle has the altitude from A in a triangle ABC as a diameter, and intersects AB and AC in the points D and E, respectively, different from A. Prove that the circumcentre of triangle ABC lies on the altitude from A in triangle ADE, or its produced.

4. A real-valued function f is defined for positive integers, and a positive integer a satifies
$$f(a) = f(1995), \quad f(a+1) = f(1996), \quad f(a+2) = f(1997),$$
$$f(n+a) = \frac{f(n)-1}{f(n)+1} \text{ for any positive integer } n.$$
   (a) Prove that $f(n+4a) = f(n)$ for any positive integer n.
   (b) Determine the smallest possible value of a.

Time allowed: 4 hours.

Each problem is valid 5 points.