18th Nordic Mathematical Contest

Thursday April 1, 2004

Time allowed: 4 hours. Each problem is worth 5 points.

Problem 1

27 balls numerated from 1 to 27 are distributed in a red, a blue and a yellow bowl. Which are the possible values of the number of balls in the red bowl if the average label of the balls in the red, blue and yellow bowl is 15, 3 and 18, respectively?

Problem 2

Let $f_1 = 0$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n = 1, 2, \ldots$, be the sequence of the Fibonacci numbers. Show there exists a (strictly) increasing arithmetic sequence of integers that has no number in common with the Fibonacci sequence.

[A sequence is *arithmetic* if the difference between successive terms is constant.]

Problem 3

Let x_{11}, x_{21}, $\ldots$, x_{n1}, $n > 2$, be a sequence of integers, and assume that the numbers x_{i1} are not all equal. Assuming the sequence x_{1k}, x_{2k}, $\ldots$, x_{nk} to be defined, set

$$x_{i,k+1} = \frac{1}{2}(x_{ik} + x_{i+1,k}),\ i = 1, 2, \ldots, n-1, \quad x_{n,k+1} = \frac{1}{2}(x_{nk} + x_{1k}).$$

If n is odd, show that for some j, k, x_{jk} is not an integer. Is this also true if n is even?

Problem 4

Let a, b and c be the sides of a triangle and let R be the radius of the circumcircle. Show that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{1}{R^2}.$$

The only instruments allowed are writing and drawing tools.