BALTIC WAY 2015
Allotted time: 9.00–13.30
Questions may be asked 9.00–9.30.
Only writing materials allowed.

1. For $n \geq 2$, an equilateral triangle is divided into n^2 congruent smaller equilateral triangles. Determine all ways in which real numbers can be assigned to the $\frac{(n+1)(n+2)}{2}$ vertices so that three such numbers sum to zero whenever the three vertices form a triangle with edges parallel to the sides of the big triangle.

2. Let n be a positive integer and let $a_1, \ldots, a_n$ be real numbers satisfying $0 \leq a_i \leq 1$ for $i = 1, \ldots, n$. Prove the inequality
$$(1 - a_1^n)(1 - a_2^n) \cdots (1 - a_n^n) \leq (1 - a_1 a_2 \cdots a_n)^n.$$

3. Let $n > 1$ be an integer. Find all non-constant real polynomials $P(x)$ satisfying, for any real x, the identity
$$P(x)P(x^2)P(x^3) \cdots P(x^n) = P\left(x^{\frac{n(n+1)}{2}}\right).$$

4. A family wears clothes of three colours: red, blue and green, with a separate, identical laundry bin for each colour. At the beginning of the first week, all bins are empty. Each week, the family generates a total of 10 kg of laundry (the proportion of each colour is subject to variation). The laundry is sorted by colour and placed in the bins. Next, the heaviest bin (only one of them, if there are several that are heaviest) is emptied and its contents washed. What is the minimal possible storing capacity required of the laundry bins in order for them never to overflow?

5. Find all functions $f\colon \mathbf{R} \to \mathbf{R}$ satisfying the equation
$$|x| f(y) + y f(x) = f(xy) + f(x^2) + f(f(y))$$
for all real numbers x and y.

6. Two players play the following game. At the outset there are two piles, containing $10,000$ and $20,000$ tokens, respectively. A move consists of removing any positive number of tokens from a single pile *or* removing $x > 0$ tokens from one pile and $y > 0$ tokens from the other, where $x + y$ is divisible by 2015. The player who cannot make a move loses. Which player has a winning strategy?

7. There are 100 members in a ladies' club. Each lady has had tea (in private) with exactly 56 of her lady friends. The Board, consisting of the 50 most distinguished ladies, have all had tea with one another. Prove that the entire club may be split into two groups in such a way that, within each group, any lady has had tea with any other.

8. With inspiration drawn from the rectilinear network of streets in New York, the *Manhattan distance* between two points (a, b) and (c, d) in the plane is defined to be
$$|a - c| + |b - d|.$$
Suppose only two distinct Manhattan distances occur between all pairs of distinct points of some point set. What is the maximal number of points in such a set?

9. Let $n > 2$ be an integer. A deck contains $\frac{n(n-1)}{2}$ cards, numbered
$$1,\ 2,\ 3,\ \ldots,\ \frac{n(n-1)}{2}.$$
Two cards form a *magic pair* if their numbers are consecutive, or if their numbers are 1 and $\frac{n(n-1)}{2}$.
For which n is it possible to distribute the cards into n stacks in such a manner that, among the cards in any two stacks, there is exactly one magic pair?

10. A subset S of $\{1,2,\ldots,n\}$ is called *balanced* if for every $a \in S$ there exists some $b \in S$, $b \neq a$, such that $\frac{a+b}{2} \in S$ as well.

    (a) Let $k > 1$ be an integer and let $n = 2^k$. Show that every subset S of $\{1,2,\ldots,n\}$ with $|S| > \frac{3n}{4}$ is balanced.

    (b) Does there exist an $n = 2^k$, with $k > 1$ an integer, for which every subset S of $\{1,2,\ldots,n\}$ with $|S| > \frac{2n}{3}$ is balanced?

11. The diagonals of the parallelogram $ABCD$ intersect at E. The bisectors of $\angle DAE$ and $\angle EBC$ intersect at F. Assume that $ECFD$ is a parallelogram. Determine the ratio $AB : AD$.

12. A circle passes through vertex B of the triangle ABC, intersects its sides AB and BC at points K and L, respectively, and touches the side AC at its midpoint M. The point N on the arc BL (which does not contain K) is such that $\angle LKN = \angle ACB$. Find $\angle BAC$ given that the triangle CKN is equilateral.

13. Let D be the footpoint of the altitude from B in the triangle ABC, where $AB = 1$. The incentre of triangle BCD coincides with the centroid of triangle ABC. Find the lengths of AC and BC.

14. In the non-isosceles triangle ABC the altitude from A meets side BC in D. Let M be the midpoint of BC and let N be the reflection of M in D. The circumcircle of the triangle AMN intersects the side AB in $P \neq A$ and the side AC in $Q \neq A$. Prove that AN, BQ and CP are concurrent.

15. In triangle ABC, the interior and exterior angle bisectors of $\angle BAC$ intersect the line BC in D and E, respectively. Let F be the second point of intersection of the line AD with the circumcircle of the triangle ABC. Let O be the circumcentre of the triangle ABC and let D' be the reflection of D in O. Prove that $\angle D'FE = 90°$.

16. Denote by $P(n)$ the greatest prime divisor of n. Find all integers $n \geq 2$ for which

$$P(n) + \lfloor \sqrt{n} \rfloor = P(n+1) + \lfloor \sqrt{n+1} \rfloor.$$

    (Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

17. Find all positive integers n for which $n^{n-1} - 1$ is divisible by 2^{2015}, but not by 2^{2016}.

18. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 1$ with n (not necessarily distinct) integer roots. Assume that there exist distinct primes $p_0, p_1, \ldots, p_{n-1}$ such that $a_i > 1$ is a power of p_i, for all $i = 0, \ldots, n-1$. Find all possible values of n.

19. Three pairwise distinct positive integers a, b, c, with $\gcd(a,b,c) = 1$, satisfy

$$a \mid (b-c)^2, \qquad b \mid (c-a)^2 \qquad \text{and} \qquad c \mid (a-b)^2.$$

    Prove that there does not exist a non-degenerate triangle with side lengths a, b, c.

20. For any integer $n \geq 2$, we define A_n to be the number of positive integers m with the following property: the distance from n to the nearest multiple of m is equal to the distance from n^3 to the nearest multiple of m. Find all integers $n \geq 2$ for which A_n is odd.

    (Note: The distance between two integers a and b is defined as $|a - b|$.)